Quasi-symmetric Functions in Geometry

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Symmetric polynomials:

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$$\mathbb{Z}[x_1, ..., x_n] \ \supset \ \mathrm{QSym}_{\mathbb{Z}}(x_1, ..., x_n) \ \supset \ \mathrm{Sym}_{\mathbb{Z}}(x_1, ..., x_n)$$

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$$f(x_1, x_2) = C + x_1\, g(x_1) + x_2\, g(x_2) + x_1\, x_2 \cdot h(x_1, x_2)$$

Can extend to bounded degree power-series in countably many variables to get quasi-symmetric *functions*:

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- Dhruv

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Well-behaved multiplication

$$\mathrm{QSym} \otimes \mathrm{QSym} \to \mathrm{QSym}$$

Well-behaved comultiplication

$$\mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$$

An antipodal map

$$\mathrm{QSym} \to \mathrm{QSym}$$

Hopf algebras occur in nature

- Algebras over groups
- Universal enveloping algebra of a lie algebra
- Cohomology of lie groups

‘Nice’ geometric space’	$\dashrightarrow$	Chow Ring
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$\mathcal{X}$ Toric Stack		$A^*(\mathcal{X})$ Cohomology

Theorem

There is a natural isomorphism

$$QSym \quad \cong \quad A^*(\, X_{\widehat{\sigma}_\infty} \,),$$

where $X_{\widehat{\sigma}_\infty}$ *is the toric stack obtained from the moduli space* $\widehat{\sigma}_\infty$ *of finitely many points on* $\mathbb{R}_{\geq 0}$.

Algebra

Rings and ideals

$$y^2 - x(x+1)(x-1) \quad \dashrightarrow \quad S^1 \times S^1$$

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Cones and fans

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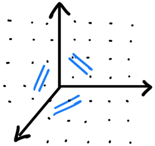

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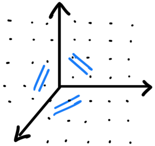

$$\dashrightarrow \mathbb{P}^2$$

- × Ugly equations.
- × Need to read ∞ pages of Hartshorne before understanding cohomology.
- × Requires brain power to do anything with.

- ✓ Pretty pictures.
- ✓ Chow ring is just the ring of piecewise polynomial functions on the fan.
- ✓ Parth and Dan can do a summer project with this.

Let Σ be the fan, X_Σ be the associated toric stack. Then

$$\mathrm{PP}^*(\Sigma) \quad \cong \quad A^*(X_\Sigma)$$

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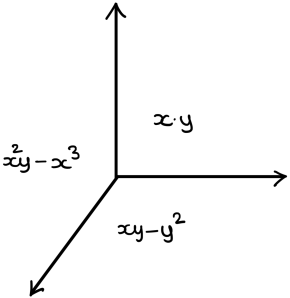

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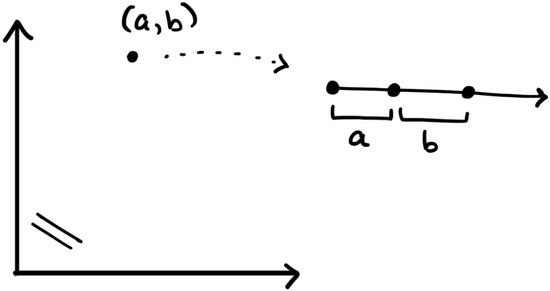

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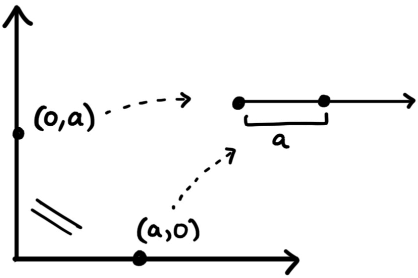
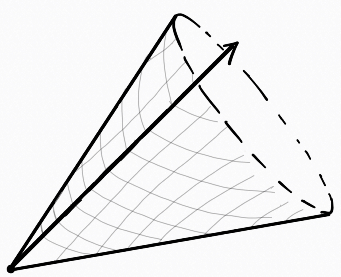


$$\widehat{\sigma}_2 \quad \cong \quad \mathbb{R}^2_{\geq 0}/((a,0) \sim (0,a)) \quad \cong$$



So we have

$$\mathrm{PP}^*(\widehat{\sigma}_2) \quad = \quad \{f(x_1, x_2) \mid f(a, 0) = f(0, a)\}.$$

By Euclid's algorithm,

$$f(x_1, x_2) = C + x_1 \cdot p(x_1) + x_2 \cdot q(x_2) + x_1 \cdot x_2 \cdot r(x_1, x_2)$$

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which occurs if and only if

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or in other words

$$f(x_1, x_2) \in \mathrm{QSym}(x_1, x_2).$$

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This readily generalises:

$$\mathrm{PP}^*(\widehat{\sigma}_\infty) \quad \cong \quad \mathrm{QSym}(x_1, x_2, x_3, \ldots).$$

Here $\widehat{\sigma}_\infty$ is the moduli space of finitely many points in $\mathbb{R}_{\geq 0}$, constructed via

$$\begin{array}{rcl} \mathbb{R}^\infty_{\geq 0} & \twoheadrightarrow & \widehat{\sigma}_\infty \\ (a_1, a_2, a_3, \ldots) & \mapsto & \{a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3, \ldots\}. \end{array}$$

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This data constructs a toric stack $X_{\widehat{\sigma}_\infty}$.

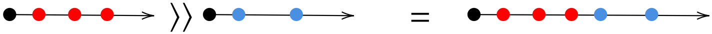
Theorem

There are natural isomorphisms

$$QSym \quad \cong \quad PP^*(\widehat{\sigma}_\infty) \quad \cong \quad A^*(\, X_{\widehat{\sigma}_\infty} \,),$$

where $X_{\widehat{\sigma}_\infty}$ *is the toric stack obtained from the moduli space* $\widehat{\sigma}_\infty$ *of finitely many points on* $\mathbb{R}_{\geq 0}$*.*

The stack has a natural binary operation:



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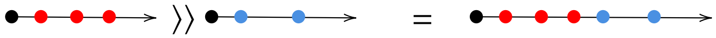

and the diagonal map $x \mapsto (x, x)$.

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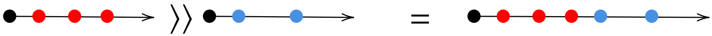

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The operations lift:

$$\widehat{\sigma}_\infty \times \widehat{\sigma}_\infty \to \widehat{\sigma}_\infty \quad \dashrightarrow \quad \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}$$

$$\widehat{\sigma}_\infty \to \widehat{\sigma}_\infty \times \widehat{\sigma}_\infty \quad \dashrightarrow \quad \mathrm{QSym} \otimes \mathrm{QSym} \to \mathrm{QSym}$$

In the rest of the project, we worked towards various generalisations of this– in particular to the moduli space of points in $\mathbb{R}^2_{\geq 0}$.