Kuznetsov components of Fano varieties

Based on a Lecture series by Arend Bayer.

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We will discuss methods to study Kuznetsov components of derived categories of coherent sheaves, using moduli theory and stability spaces. Along the way we will look at classical questions regarding invariants and techniques surrounding Fano varieties.

§ 0.1 About. This course on Fano varieties was a part of a Winter School for students and early career researchers, organised by the UK Algebraic Geometry Network. The lectures were delivered in-person in the University of Warwick, and were transcribed by Parth Shimpi. These notes have undergone several amendments and are not a verbatim recall of the lectures, therefore discretion is advised when using this material. They are available online at `https://pas201.user.srcf.net/documents/2023-ukag-kuznetsov.pdf`. All errors and corrections should be communicated to by email to `parth.shimpi@glasgow.ac.uk`.

§ 1 Low-dimensional Fano varieties

Definition 1.1. A smooth variety $X/\mathbb{C}$ is called *Fano* if the anticanonical bundle $\mathcal{O}(-K_X) = \bigwedge^{\mathrm{top}} T_X$ is ample.

Thus the only one-dimensional Fano variety is $\mathbb{P}^1$.

More generally, all projective spaces and complete intersections of low degree in projective spaces are Fano. In particular, any complete intersection $X \subset \mathbb{P}^{n+N}$ cut out by equations of degrees $d_1, \ldots, d_N$ is Fano if and only if $\sum_i d_i \leqslant n + N$. For example, this can be used to show that all Fano 2-folds (i.e. del Pezzo surfaces) arise as $\mathbb{P}^2$ blown up at $\leqslant 8$ points, or as the smooth quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

Example 1.2 (Grassmannians). Generalising the example of projective space, the Grassmannian $G = \mathrm{Gr}(n, m)$ parametrising n-dimensional subspaces $U_n \subset \mathbb{C}^m$ is Fano. To see why, we will compute its tangent bundle. Note we have the tautological exact sequence of vector bundles $\mathcal{U}_n \hookrightarrow \mathcal{O}_X^{\oplus m} \twoheadrightarrow \mathcal{Q}_{m-n}$, and that at a point $p \in X$ corresponding to $U_n \hookrightarrow \mathbb{C}^m \twoheadrightarrow Q_{m-n}$ the tangent space T_pX is given by elements of $\mathrm{Hom}(U_n, Q_{m-n})$ i.e. deformations of the subspace $U_n \hookrightarrow \mathbb{C}^m$ up to automorphisms of U_n. Thus the tangent bundle is $T_X = \mathrm{Hom}(\mathcal{U}_n, \mathcal{Q}_{m-n})$, and we have $\bigwedge^{\mathrm{top}} T_X = (\bigwedge^{\mathrm{top}} \mathcal{U}_n^\vee)^{\otimes(m-n)} \otimes (\bigwedge^{\mathrm{top}} \mathcal{Q}_{m-n})^{\otimes n}$. But the tautological exact sequence shows $\bigwedge^{\mathrm{top}} \mathcal{U}_n = (\bigwedge^{\mathrm{top}} \mathcal{Q}_{m-n})^{-1}$, and the Plücker embedding $G \hookrightarrow \mathbb{P}(\bigwedge^n \mathbb{C}^m)$ shows $\bigwedge^{\mathrm{top}} \mathcal{U}_n = \mathcal{O}(1)$ is ample. We conclude that so is $\bigwedge^{\mathrm{top}} T_X = \mathcal{O}_X(m)$.

Likewise, low degree subvarieties of Grassmannians are Fano, and this gives us examples of Fano varieties in dimensions three and four. For example, $\mathrm{Gr}(2,4) \subset \mathbb{P}^5$ is a smooth quadric 4-fold. Intersecting $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ with three hyperplanes, we get the Fano 3-fold $Y_5 = \mathrm{Gr}(2,5) \cap \mathbb{P}^6$. Likewise we have the *Gushel–Mukai 3-fold* X_{10} obtained by intersecting $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ with two hyperplanes and a quadric. Another similar example is $X_{14} = \mathrm{Gr}(2,6) \cap \mathbb{P}^9 \subset \mathbb{P}^{14}$.

§ 1.1 Deformation families. Fano varieties of fixed dimension are *bounded*, i.e. occur in finitely many deformation families. In low dimensions, there are fairly explicit unirational moduli spaces while the moduli spaces in higher dimensions are studied using K-stability.

For Fano 3-folds, there are 105 families. Most have relatively low Picard rank– this is because the behaviour is fairly complicated for high Picard ranks and in fact sufficiently high Picard ranks simply become difficult to realise. For instance the only way to obtain a Fano 3-fold of Picard rank 8 is the trivial way, by taking the product of $\mathbb{P}^1$ with a del Pezzo surface.

Thus we focus on the simplest case, that of Picard rank 1. Such Fano 3-folds are called *prime*, and have a canonical ample divisor H (called the *fundamental divisor*) which generates $\mathrm{Pic}\, X$. We say the *index* of X is the integer i_X satisfying $K_X + i_X \cdot H \equiv 0$, and the *degree* of X is the self-intersection $d_X = -K_X^3$. It is also convenient to introduce the number $g_X = \frac{1}{2}(d_X + 2)$, called the *genus* (the reason for this terminology will be explained later.)

Remark 1.3. If X is a prime Fano 3-fold, then we must have $1 \leqslant i_X \leqslant 4$. Indeed the lower bound comes from the Fano condition. To find the upper bound, note that if $S \in |H|$ is a general element in the linear system then $K_S = -(i_X - 1)H|_S$ by adjunction. In particular if $i_X > 1$ then S is a del Pezzo surface and we have the well-known bound

$$1 \leqslant K_S^2 = (i_X - 1)^2 H^3 \leqslant 9.$$

which shows i_X cannot exceed 4. Note $i_X = 4$ forces $H^3 = 1$, and the only such Fano 3-fold is $\mathbb{P}^3$.

The deformation type of X is determined by i_X and d_X, so it suffices to compute the possible values of d_X for each $i_X \in \{1, 2, 3, 4\}$. For $i_X = 3$ the only possibility is $H^3 = 2$ i.e. $d_X = 54$ ($H^3 = 1$ is ruled out since the genus is an integer) and the variety is a quadric in $\mathbb{P}^4$.

For $i_X = 2$ the above remark shows $1 \leqslant d_X \leqslant 9$, but in fact only $1 \leqslant d_X \leqslant 5$ are realised. These are given by Y_d ($1 \leqslant d \leqslant 5$), where Y_5 was described above. $Y_4 \subset \mathbb{P}^5$ is a $(2, 2)$ complete intersection, $Y_3 \subset \mathbb{P}^4$ is a cubic 3-fold, and $Y_2 \twoheadrightarrow \mathbb{P}^3$ is a double cover.

For $i_X = 1$, we have the following result.

Theorem 1.4 (Ishkowskii). *If* X *is a prime Fano 3-fold of index 1, then we have* $2 \leqslant g_X \leqslant 12$ *and* $g_X \neq 11$. *This is called the* Fano range.

Thus there are seventeen families of prime Fano 3-folds.

§1.2 A closer look at index 1. There is a beautiful correspondence between Fano 3-folds of index 1, K3 surfaces, and curves which was first shown by Mukai. Indeed in this case the anticanonical bundle $-K_X = H$ is a natural polarisation, and a general element in the linear system $|H|$ is a smooth K3 surface S.

Moreover this surface naturally comes with the polarisation $H|_S$, and a general element in this linear system is a genus g_S curve (the genus of a polarised K3 surface is defined this way). Then it can be shown that g_S coincides with g_X.

Mukai showed that general K3 surfaces arise in this way.

Theorem 1.5 (Mukai). *If* g *is in the Fano range, then a general polarised K3 surface of genus* g *arises as a hyperplane section of a Fano 3-fold. Moreover, a general canonical curve arises as a hyperplane section of a polarised K3 (and hence as a complete intersection of a Fano 3-fold).*

§1.3 Irrationality of the cubic 3-fold. A discussion on low dimensional Fano varieties would not be complete without mentioning this famous result. If X is a Fano 3-fold, then $H^{3,0}(X) = H^0(X, \mathcal{O}_X(K_X)) = 0$ and hence we can consider $H^3(X, \mathbb{Z}) \subset H^3(X, \mathbb{C}) = H^{2,1}(X) \oplus H^{1,2}(X)$. The space $I\mathcal{J}ac X = H^{2,1}(X, \mathbb{C})^\vee / H_3(X, \mathbb{Z})$ is a polarised Abelian variety called the *intermediate Jacobian of* X.

The behaviour of this under birational transformations is well-understood. Firstly it is invariant under blowing up in codimension 2 locii, i.e. we have $I\mathcal{J}ac(\mathrm{Bl}_{\mathrm{pts}} X) = I\mathcal{J}ac X$. On the other hand if $C \subset X$ is a curve, then $I\mathcal{J}ac(\mathrm{Bl}_C X) = I\mathcal{J}ac X \times \mathcal{J}ac C$. Since Jacobians of curves are irreducible as Abelian varieties and $\mathbb{P}^2$ has trivial intermediate Jacobian, we use weak factorisation to deduce that $I\mathcal{J}ac X$ is a product of Jacobians whenever X is rational. This is used to deduce the following.

Theorem 1.6 (Clemens–Griffiths). *A general cubic 3-fold is irrational.*

Sketch. If X is a cubic 3-fold, then $I\mathcal{J}ac X$ is a five dimensional principally polarised Abelian variety. We compute that the theta divisor $\Theta \subset$ IJacX has minimal degree and an isolated singularity x_0. This can happen only if $I\mathcal{J}ac X$ is indecomposable and not a Jacobian, and hence X is not rational. ☐

Similar arguments also give Torelli theorem for cubic 3-folds, by showing the projectivised tangent cone $\mathbb{P}T_{x_0}\Theta$ is isomorphic to X. Thus X is determined by the principally polarised Abelian variety $(I\mathcal{J}ac X, \Theta)$, which is in turn determined by the polarised Hodge structure on $H^3(X, \mathbb{C})$.

§2 Semiorthogonal decompositions

Given a variety $X/\mathbb{C}$, recall the construction of the derived category of coherent sheaves $\mathbf{D}(X) = \mathbf{D}^b \mathcal{C}oh X$. If X is Fano, then a classical result of Bondal–Orlov shows that $\mathbf{D}(X)$ determines X. We will refine this result and ask if X is determined by certain canonical subcategories.

Definition 2.1. An object $E \in \mathbf{D}(X)$ is called *exceptional* if $\mathrm{Hom}^\bullet(E, E) = \mathbb{C}[0]$. An *exceptional collection* is an ordered sequence $(E_1, \ldots, E_n)$ of exceptional objects such that $\mathrm{Hom}^\bullet(E_i, E_j) = 0$ whenever $i > j$. We say an exceptional collection is *full* if it generates $\mathbf{D}(X)$ (i.e. is not contained in a proper triangulated subcategory).

Fano varieties are naturally equipped with exceptional collection. For example, a classical result of Beilinson shows $\mathbb{P}^n$ is has a full exceptional collection $(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1), \dots, \mathcal{O}_{\mathbb{P}^n}(n))$. This generalises to the following result.

Lemma 2.2. *If X is Fano and $L \in D(X)$ is a line bundle, then L is exceptional. Moreover, if L is an ample line bundle such that $\mathcal{O}(-K_X) = L^{\otimes r}$ then $(\mathcal{O}_X, L, \dots, L^{\otimes r-1})$ is an exceptional collection.*

Proof. We will show the second part of the statement, the first is analogous. We need to compute the group $\mathrm{Ext}^q(L^{\otimes i}, L^{\otimes j})$ for $r > i \geqslant j \geqslant 0$. This is equal to $H^q(X, \mathcal{O}_X(-K_X) \otimes L^{\otimes r+j-i})$. We know $r + j - i > 0$ and hence $L^{\otimes r+j-i}$ is ample. Thus the required cohomology vanishes for $q > 0$ by the Kawamata–Veibig vanishing theorem. On the other hand if $q = 0$ and $i > j$ then the group is $H^0(X, L^{\otimes j-i})$ which vanishes since anti-ample bundles don't have global sections. □

Weakening the definition of a full exceptional collection, we have the following.

Definition 2.3. If $\mathcal{D}$ is a triangulated category, then a *semi-orthogonal decomposition* denoted $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2, \dots, \mathcal{D}_n \rangle$ is a sequence of full triangulated subcategories that generate $\mathcal{D}$ and satisfy $\mathcal{D}_i \subset {\mathcal{D}_j}^{\perp}$ whenever $i < j$.

In particular, two-term semorthogonal decompositions $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ are determined by two additive subcategories $\mathcal{D}_1, \mathcal{D}_2$ that are closed unders shifts and such that every $E \in \mathcal{D}$ fits in an exact triangle $E_2 \to E \to E_1 \dashrightarrow$ for $E_1 \in \mathcal{D}_1$, $E_2 \in \mathcal{D}_2$. Finding semi-orthogonal decompositions of $\mathcal{D}_1$ and $\mathcal{D}_2$ and iterating gives semi-orthogonal decompositions of $\mathcal{D}$ with multiple terms.

Thus, for example, every partition of a full exceptional collection gives a semi-orthogonal decomposition. If L is an exceptional object, then the corresponding component $\langle L \rangle$ is equivalent to the derived category of a point (since every object has form $L \otimes V^\bullet$ for a complex of vector spaces $V^\bullet$).

Example 2.4. The category $\mathbf{D}(\mathbb{P}^1)$ is "built out of two points", in the sense that there is a full exceptional collection $(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-1))$. Indeed we have already seen the orthogonality condition, and if $E \in \mathbf{D}(\mathbb{P}^1)$ is indecomposable then it is either of the form $\mathcal{O}_{\mathbb{P}^1}(k)[i]$ for some $k, i \in \mathbb{Z}$ or of the form $\mathcal{O}_x[i]$ for some skyscraper sheaf $\mathcal{O}_x$ and $i \in \mathbb{Z}$. The existence of the required exact triangle can then be deduced from triangles of the form $\mathcal{O}_{\mathbb{P}^1}^{\oplus k-1} \to \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus k} \to \mathcal{O}_{\mathbb{P}^1}(k) \dashrightarrow$ and $\mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_x \to \mathcal{O}_{\mathbb{P}^1}(-1)[1] \dashrightarrow$.

§ 2.1 Admissible subcategories. Semi-orthogonal decompositions can be determined from just one component, hence it suffices to look for certain well-behaved subcategories.

Definition 2.5. A full subcategory $\mathcal{D}' \subset \mathcal{D}$ is *left (right) admissible* if the inclusion functor has a left (resp. right) adjoint. We say $\mathcal{D}'$ is *admissible* if it is both left and right admissible.

Proposition 2.6. *If $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is a semi-orthogonal decomposition, then $\mathcal{D}_1$ is left admissible and $\mathcal{D}_2$ is right admissible. Conversely if $\mathcal{D}_1 \subset \mathcal{D}$ is a left admissible subcategory then $\langle \mathcal{D}_1, {}^{\perp}\mathcal{D}_1 \rangle$ is a semi-orthogonal decomposition. Likewise for right admissible subcategories $\mathcal{D}_2 \subset \mathcal{D}$.*

Proof. If $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ is a semi-orthogonal decomposition then every $E \in \mathcal{D}$ sits in a unique(!) triangle $E_2 \to E \to E_1 \dashrightarrow$ with $E_i \in \mathcal{D}_i$. Then the required adjoint functors are given by $E \mapsto E_i$. Conversely if the inclusion $i : \mathcal{D}_2 \to \mathcal{D}$ has a right adjoint $i^! : \mathcal{D} \to \mathcal{D}_2$, then for any $E \in \mathcal{D}$ we have the adjunction map $E \to i \circ i^! E$ with cone E_1. Remains to show $E_1 \in {\mathcal{D}_2}^{\perp}$, which holds since for any $F \in \mathcal{D}_2$ we have

$$\mathrm{Hom}^\bullet(iF, E) = \mathrm{Hom}^\bullet(F, i^! E) = \mathrm{Hom}^\bullet(iF, i \circ i^! E)$$

i.e. the term $\mathrm{Hom}^\bullet(iF, E_1)$ must vanish in the long exact sequence associated to $E \to i \circ i^! E \to E_1 \dashrightarrow$. □

Remark 2.7. If $\mathcal{D}$ has a Serre functor $\mathbb{S}$ (e.g. if $\mathcal{D} = D(X)$ for X smooth) then semi-orthogonal decompositions always come in pairs $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle = \langle \mathbb{S}\mathcal{D}_2, \mathcal{D}_1 \rangle$. In particular a category is left admissible if and only if it is right admissible. Now if $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3 \rangle$ is a semi-orthogonal decomposition such that the subcategory $\langle \mathcal{D}_2, \mathcal{D}_3 \rangle$ has a Serre functor $\mathbb{S}_{23}$, then we obtain a new semi-orthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \mathbb{S}_{23}\mathcal{D}_3, \mathcal{D}_2 \rangle$. This general process of *mutation* can be seen as an action of a braid group on the semi-orthogonal decomposition.

Where to find admissible subcategories? The following theorem gives one source.

Theorem 2.8. *If X, Y are smooth varieties and $\Phi : D(X) \to D(Y)$ is fully faithful, then the image of Φ is admissible.*

Exceptional objects can be seen as a special instance of the above. Note that $E \in \mathcal{D}$ is an exceptional object if and only if the map $D(\mathrm{pt}) \to \mathcal{D}$ given by $V^\bullet \mapsto E \otimes V^\bullet$ is fully faithful. The image is right admissible as $F \mapsto \mathrm{Hom}(E, F)$ is the right adjoint by the tensor–hom adjunction.

Thus exceptional object induces a semi-orthogonal decomposition $\mathcal{D} = \langle E^\perp, E\rangle$. Iterating this construction, we can induce semi-orthogonal decompositions from exceptional collections.

Definition 2.9. If $E_1, ..., E_m$ is an exceptional collection in $D(X)$ then the *Kuznetsov component* of $D(X)$ with respect to this exceptional collection is $\mathrm{Ku}(X) = {E_1}^\perp \cap {E_2}^\perp \cap ... \cap {E_m}^\perp$. Thus $D(X) = \langle \mathrm{Ku}(X), E_1, ..., E_m\rangle$ is a semi-orthogonal decomposition.

While the definition of a Kuznetsov component depends on the choice of an exceptional collection, such a choice is often canonically made. We will see some examples.

§ 2.2 Semi-orthogonal decompositions on families. Take a quadric $Q \subset \mathbb{P}^3$. If Q is smooth then it is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so there is a full exceptional collection $(\mathcal{O}_Q(-1,0), \mathcal{O}_Q(0,-1), \mathcal{O}_Q, \mathcal{O}_Q(1,1))$. Of these, the latter two are canonical (they come from the structure sheaf and the canonical bundle) so we choose those as our exceptional objects and hence we have the Kuznetsov component $\mathrm{Ku}(Q) = \langle \mathcal{O}_Q(-1,0), \mathcal{O}_Q(0,-1)\rangle = D(\mathrm{pt})^{\oplus 2}$.

If Q is nodal singular (i.e. given by $\mathbb{V}(xy - z^2) \subset \mathbb{P}^3$), then the Kuznetsov component is generated by the ideal sheaf of the line $\ell = \mathbb{V}(x, z)$ so we have $\mathrm{Ku}(Q) = D(k[\epsilon]/(\epsilon^2))$. The intuition is that as the quadric surface degenerates from smooth to singular, the Kuznetsov component undergoes a similar transition. To formalise this, we will study families of varieties.

Definition 2.10. Let $\pi : X \to B$ be a Gorenstein morphism. A *relative exceptional object* is a perfect complex $E \in \mathbf{D}(X)$ such that $\mathcal{H}om_B^\bullet(E, E) := \pi_* \mathcal{H}om^\bullet(E, E) \cong \mathcal{O}_B[0]$. A *relative exceptional collection* is a sequence of relative exceptional objects $E_1, ..., E_m$ with $\mathcal{H}om_B(E_i, E_j) = 0$ for $i > j$.

The Gorenstein assumption guarantees the Serre functor takes bounded complexes to bounded complexes, and the assertion that E is perfect (i.e. quasi-isomorphic to a bounded complex of vector bundles) guarantees that it is well-behaved under tensor products. It follows that the map $\pi^*(-) \otimes E : \mathbf{D}(B) \to \mathbf{D}(X)$ is fully faithful with admissible image, so we can induce semi-orthogonal decompositions as before.

Going back to quadrics, let $\pi : \mathcal{Q} \to B$ be a family of quadric surfaces with $\dim B = 1$, and $\mathcal{Q}, B$ smooth. It follows that the fibers are at worst nodal. Then $(\mathcal{O}_\mathcal{Q}, \mathcal{O}_\mathcal{Q}(1))$ is a relative exceptional collection giving us a semi-orthogonal decomposition

$$\mathbf{D}(\mathcal{Q}) = \langle \mathrm{Ku}(\mathcal{Q}), \pi^* \mathbf{D}(B), \pi^* \mathbf{D}(B) \otimes \mathcal{O}_\mathcal{Q}(1)\rangle .$$

To understand the Kuznetsov component better, consider the 2:1 cover associated to π obtained as follows. We have the relative Hilbert scheme $\mathrm{Hilb}_{\mathrm{lines}}(\mathcal{Q}) \to B$, with Stein factorisation $\mathrm{Hilb}_{\mathrm{lines}}(\mathcal{Q}) \to \tilde{B} \to B$. Thus the map $\mathrm{Hilb}_{\mathrm{lines}}(\mathcal{Q}) \to \tilde{B}$ is proper with connected fibers, while the map $\tilde{B} \to B$ is a $2:1$ cover branched at the discriminant locus in B.

We can think of $\tilde{B}$ is the relative moduli space of torsion-free sheaves F in the fibers of π with an equality of Hilbert polynomials $p_H(F) = p_H(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1,0))$. As a result there is a universal family in $\mathcal{C}oh(\tilde{B} \times \mathcal{Q})$, giving a Fourier–Mukai transform $\Phi : \mathbf{D}(\tilde{B}) \to \mathbf{D}(\mathcal{Q})$. It can be shown that this is fully faithful with image $\mathrm{Ku}(\mathcal{Q})$, so that the semi-orthogonal decomposition is

$$\mathbf{D}(\mathcal{Q}) = \langle \mathbf{D}(\tilde{B}), \pi^* \mathbf{D}(B), \pi^* \mathbf{D}(B) \otimes \mathcal{O}_\mathcal{Q}(1)\rangle.$$

Since relative exceptional collections are well-behaved under pullbacks, this explains the relation between the Kuznetsov components of smooth and nodal quadrics– if the fiber of $\mathcal{Q}$ over $b \in B$ is smooth, the moduli space $\tilde{B}$ has two points corresponding to $\mathcal{O}(-1,0)$ and $\mathcal{O}(0,-1)$. On the other hand $\tilde{B} \to B$ is branched precisely over the $b \in B$ corresponding to nodal quadrics, where the fiber of $\tilde{B}$ looks like $\operatorname{Spec} k[\epsilon]/(\epsilon^2)$.

§ 2.3 Decompositions via birational transformations. Orlov's blow-up formula explains how Kuznetsov components transform under birational maps– in general blowing up along a codimension r locus $Z \hookrightarrow X$ adds $r-1$ copies of $\mathbf{D}(Z)$ twisted by some exceptional bundles to $\mathbf{D}(\mathrm{Bl}_Z X)$.

More specifically if X is a 3-fold with a smooth point $x \in X$, then the blow-up $p : \mathrm{Bl}_x X \to X$ has exceptional fiber $E \cong \mathbb{P}^2$ and we have $\mathbf{D}(\mathrm{Bl}_x X) = \langle \mathcal{O}_E(-2), \mathcal{O}_E(-1), p^* \mathbf{D}(X)\rangle$. In particular, the Kuznetsov component of X is "essentially undamaged", it simply picks up two points. On the other hand if $C \subset X$ is a smooth curve then we have $\mathbf{D}(\mathrm{Bl}_C X) = \langle \mathbf{D}(C), p^* \mathbf{D}(X)\rangle$ i.e. the Kuznetsov component picks up $\mathbf{D}(C)$. Note the striking similarity with intermediate Jacobians!

To see a concrete example, fix two quadrics $Q_1, Q_2 \subset \mathbb{P}^3$ and consider the associated pencil. This gives a family $\mathcal{Q}$ over $B = \mathbb{P}^1$, which can be realised as a blow-up $p : \mathcal{Q} \to \mathbb{P}^3$ along the elliptic curve $C = Q_1 \cap Q_2$ giving us

$$\mathbf{D}(\mathcal{Q}) = \langle \mathbf{D}(C), p^*\mathcal{O}_{\mathbb{P}^3}, p^*\mathcal{O}_{\mathbb{P}^3}(1), p^*\mathcal{O}_{\mathbb{P}^3}(2), p^*\mathcal{O}_{\mathbb{P}^3}(3)\rangle.$$

How does this relate to the semi-orthogonal decomposition computed previously? Note the map $\pi : \mathcal{Q} \to B$ has singular fibers at four points in B, so the 2 : 1 branched cover $\tilde{B} \to B$ is an elliptic curve. In particular, both the semi-orthogonal decompositions are made of the derived category of an elliptic curve and four exceptional objects. These are in fact related and $\tilde{B} = C$, but the proof is somewhat ad hoc.

Remark 2.11. We just saw that it is possible to have multiple semi-orthogonal decompositions that are not related in an obvious way (eg. by mutation). This is quite typical! There is however a conjectural proposal by Halpern–Leistner relating semi-orthogonal decompositions that arise 'geometrically'.

Example 2.12. Playing the above game in $\mathbb{P}^5$, consider the pencil associated to two general quadrics Q_1, Q_2. The resulting $(2,2)$-complete intersection $Q_1 \cap Q_2$ is the index 2 Fano 3-fold Y_4, and the pencil can be realised as a blow-up $\mathcal{Q} = \mathrm{Bl}_{Y_4}(\mathbb{P}^5) \to \mathbb{P}^1$. Thus we again have two ways to compute $\mathbf{D}(\mathcal{Q})$– since the blow-up is along a codimension 2 locus, Orlov's blow-up formula tells us

$$D(\mathcal{Q}) = \langle \mathbf{D}(Y_4), p^* \mathbf{D}(\mathbb{P}^5) \rangle.$$

On the other hand, the pencil $\pi : \mathcal{Q} \to \mathbb{P}^1$ has singular fibers over six points, so that the associated double cover $\tilde{B}$ is a genus 2 curve. A general smooth quadric Q in the pencil has an exceptional collection $\mathcal{O}_Q, \mathcal{O}_Q(1), \mathcal{O}_Q(2), \mathcal{O}_Q(3)$ and a Kuznetsov component coming from two spinor bundles, which we can use to find a relative exceptional collection on $\tilde{Q}$. This, together with the derived category of $\tilde{B}$ give us a semi-orthogonal decomposition

$$\mathbf{D}(\tilde{Q}) = \langle \mathbf{D}(\tilde{B}), \pi^* D(\mathbb{P}^1), \ldots, \pi^* \mathbf{D}(\mathbb{P}^1) \otimes \mathcal{O}_{\tilde{Q}}(3) \rangle.$$

Note that one semi-orthogonal decomposition has six exceptional objects (coming from $\mathbf{D}(\mathbb{P}^5)$), while the other has eight! This was explained by Bondal and Orlov who recovered the two exceptional objects by computing $\mathbf{D}(Y_4) = \langle D(\tilde{B}), \mathcal{O}_{Y_4}, \mathcal{O}_{Y_4}(1) \rangle$. In fact such a formula holds for all prime Fano 3-folds of index 2.

§ 2.4 Mukai bundles. We discuss a powerful result on how to find exceptional objects.

Theorem 2.13 (Mukai). *If X is a prime Fano 3-fold of index 1 and genus $g \geqslant 6$, then for $r = 2, 3$ and $s \geqslant r$ satisfying $rs = g$ there is a unique vector bundle $\mathcal{E}_r$ of rank r and $c_1(\mathcal{E}_r) = -H$ that is exceptional. Moreover, it is stable and satisfies $H^\bullet(X, \mathcal{E}_r) = 0$. Its dual $\mathcal{E}_r^\vee$ is globally generated by $r + s$ sections, giving a map $X \to \mathrm{Gr}(r, r+s)$. This map is an embedding unless $g = 6$ and X is "special", i.e. a double cover of Y_5.*

It is hard to overstate the importance of this theorem. For instance, it leads to the classification of prime Fano 3-folds of index 1. In particular they are complete intersections for $g \leqslant 5$. For $g = 6$, they are either special or $(2,1,1)$-complete intersections in $\mathrm{Gr}(2,5)$. For $g = 8$ they are linear sections of $\mathrm{Gr}(2,6)$ and for genus 10 they are obtained as the intersection of a linear section of $\mathrm{Gr}(2,6)$ with the zero section of a vector bundle.

As a second application, we have a canonical choice of exceptional objects $\mathcal{E}_r, \mathcal{O}_X$ so the Kuznetsov component of X is defined with respect to these, and we have $D(X) = \langle \mathrm{Ku}(X), \mathcal{E}_r, \mathcal{O}_X \rangle$.

But there's a problem. There is no complete and correct proof in the literature, so a proof of this result is a work in progress [Bayer–Kuznetsov–Macri].

§ 2.5 Perry's categorical intermediate Jacobian. Kuznetsov components and intermediate Jacobians share many similar properties– an attempt to explain this is a categorification of Jacobians. An ingredient in this categorical Jacobian are Blanc's topological K-theories $K_0^{top}\mathcal{D}, K_1^{top}\mathcal{D}$ defined for a suitably enriched $\mathbb{C}$-linear category $\mathcal{D}$. When $\mathcal{D} = \mathbf{D}(X)$ this coincides with the topological K-theory $K_i^{top}\mathcal{D} = K_i^{top}X$ and we have $K_{0,1}^{top}\mathcal{D} \otimes \mathbb{Q} = H^{\mathrm{even,odd}}(X, \mathbb{Q})$.

The construction is additive on semi-orthogonal decompositions, and has an Euler pairing. There are two Chern characters– one from the usual Grothendieck group $K_0\mathcal{D} \to K_0^{top}\mathcal{D}$, and another to the Hochschild cohomology which has a canonical filtration. This can be used to show the following result.

Theorem 2.14 (Perry). *Assume $\mathcal{D} \subset \mathbf{D}(X)$ is admissible. Then $K_{0,1}^{top}(\mathcal{D}$ carries a natural integral polarised Hodge structure compatible with the above properties. Moreover if $n = \dim X$ is odd and $H^{odd}(X, \mathbb{Z}) = H^n(X, \mathbb{Z})$, then $H^n(X, \mathbb{C}) \cong K_1^{top}D(X)$ as integral Hodge structures. If additionally $H^{p,n-p}(X) = 0$ except for $p = \frac{n \pm 1}{2}$, then $H^n(X)$ coincides with $I\mathcal{J}ac X$.*

Corollary 2.15. *In the above situation, a semiorthogonal decomposition $\mathbf{D}(X) = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ gives a decomposition $I\mathcal{J}ac X = I\mathcal{J}ac\mathcal{D}_1 \times \ldots \times I\mathcal{J}ac\mathcal{D}_n$ where the Jacobians $I\mathcal{J}ac\mathcal{D}_i$ are computed from the weight 1 intermediate Hodge structure on $K_1^{top}\mathcal{D}_i$.*

§ 3 Stability conditions and moduli spaces

We will now see how to elucidate Kuznetsov components of Fano 3-folds by studying moduli spaces of Bridgeland-semistable objects.

Definition 3.1. A *stability condition* on an Abelian category $\mathscr{A}$ is an additive map $Z : K_0\mathscr{A} \to \mathbb{Z}$ (called the *central charge*) that satisfies $Z(E) \in \mathbb{H}_-$ for any $0 \neq E \in \mathscr{A}$ and has the Harder–Narasimhan property.

We explain what the Harder–Narasimhan property is. Note since Z sends the class of a non-zero object into the semi-closed upper half plane, the *phase* of an object is a well defined real number in $(0, 1]$. We say $0 \neq E \in \mathscr{A}$ is Z-semistable if all non-zero sub-objects of E have smaller phase, or equivalently a smaller slope (where the slope of E is $\mu(E) = -\frac{\operatorname{Re} Z\, E}{\operatorname{Im} Z\, E}$). Then Z satisfies the Harder–Narasimhan property if every $0 \neq E \in \mathscr{E}$ admits a filtration $0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$ such that all factors E_i/E_{i-1} are semistable and satisfy $\mu(E_1/E_0) > \mu(E_2/E_1) > \ldots > \mu(E_n/E_{n-1})$.

Example 3.2. If $\mathscr{A} = \mathscr{Coh}\, C$ for a smooth projective curve, then $Z = i \cdot \operatorname{rank} - \deg$ gives a stability condition.

We also define a *weak stability condition*, in which we allow non-zero objects E to have $Z(E) = 0$ (in which case the slope is $+\infty$).

Example 3.3. If S is a smooth projective surface and $H = [C]$ is a polarisation, then $Z = i \cdot \operatorname{rank} - H \cdot c_1$ (i.e. $Z(E) = i \cdot \operatorname{rank}(E|_C) - \deg(E|_C)$) is a weak stability function.

To extend this to triangulated categories, we define first the 'heart of a bounded t-structure'. The prime example is an Abelian category $\mathscr{A}$ as a subcategory of $\mathbf{D}^b\,\mathscr{A}$.

Definition 3.4. If $\mathscr{D}$ is a triangulated category, then a *heart* is a full additive subcategory $\mathscr{A} \subset \mathscr{D}$ such that $\mathscr{A}[k_1] \subset {}^\perp\mathscr{A}[k_2]$ whenever $k_1 > k_2$, and every $0 \neq E \in \mathscr{D}$ admits a filtration (i.e. a sequence of maps)

$$0 = E_0 \to E_1 \to \ldots \to E_n = E$$

with $\mathscr{Cone}(E_{i-1} \to E_i) \in \mathscr{A}[k_i]$ for $k_1 > k_2 > \ldots$.

To draw the analogy with stability functions on Abelian categories, observe that if $E, F \in \mathscr{A}$ are semistable objects with slopes $\mu(E) > \mu(F)$, then $\operatorname{Hom}(E, F) = 0$. Thus $\mathscr{A}$ is filtrated by 'slices' of semistable objects much like how $\mathscr{D}$ is filtrated by shifts of $\mathscr{A}$. A stability condition combines these two ideas.

Definition 3.5. A *stability condition* on $\mathscr{D}$ is given by the heart of a bounded t-structure $\mathscr{A}$ and a stability function on $\mathscr{A}$. A weak stability condition is defined analogously.

§ 3.1 Tilting of hearts. How to construct hearts? We have already seen $\mathscr{A}$ is a heart of $\mathbf{D}^b\,\mathscr{A}$. If Φ is an autoequivalence of $\mathscr{D}$, then applying Φ to existing hearts also gives new hearts. But we need a more systematic way to get new hearts, and this can be done by perturbing existing ones.

Definition 3.6. A torsion pair (T, F) in an Abelian category $\mathscr{A}$ is a pair of full additive subcategories such that $T \subset {}^\perp F$ and every $E \in \mathscr{A}$ sits in a short exact sequence $0 \to E_T \to E \to E_F \to 0$ with $E_T \in T$, $E_F \in F$.

The prime example is that of torsion sheaves and torsion-free sheaves on a smooth variety.

Torsion pairs can be thought of as a coarse filtration of $\mathscr{A}$ into two blocks. Thus for instance if $\mathscr{A}$ has a stability function, then for any $\mu \in (0, 1]$ we have an induced torsion pair $T = \mathscr{A}^{>\mu}, F = \mathscr{A}^{\leqslant\mu}$ where the torsion part contains all objects whose semistable factors have slope $> \mu$.

Proposition 3.7. *If $\mathscr{A}$ is the heart of a bounded t-structure and (T, F) is a torsion pair on $\mathscr{A}$, then the extension closure $\mathscr{A}^\sharp = \langle T, F[1]\rangle$ is the heart of a bounded t-structure. Hearts that arise in this way are called* tilts *of $\mathscr{A}$.*

Concretely, if $\mathscr{A}$ is the natural heart of $\mathbf{D}^b\,\mathscr{A}$ then $\mathscr{A}^\sharp$ contains precisely those complexes E which have $H^{-1}(E) \in F$, $H^0(E) \in T$, and $H^i(E) = 0$ otherwise. Such E can always be represented by a 2-term complex $E^{-1} \xrightarrow{d} E^0$ with $\ker d \in F$, $\operatorname{coker} d \in T$.

§ 3.2 Constructing hearts in Kuznetsov components. In general, the process of taking semi-orthogonal decompositions is not compatible with taking hearts of bounded t-structures. However, if we are dealing with a single exceptional object then hearts descend to Kuznetsov components under reasonable assumptions.

Proposition 3.8. *Suppose $\mathscr{D}$ has an exceptional object E that lies in a heart $\mathscr{A}$. Moreover, suppose $\operatorname{Hom}^q(E, F) = 0$ for all $F \in \mathscr{A}$ unless $q = 0, 1$. Then $\mathscr{A} \cap E^\perp$ is the heart of a bounded structure in $E^\perp$.*

Proof. For $F \in E^\perp$, we wish to show that each cohomology object with respect to $\mathcal{A}$ is also in $E^\perp$. Now there is a spectral sequence $E_2^{p,q} = \mathrm{Hom}^q(E, H^p_{\mathcal{A}}F) \Rightarrow \mathrm{Hom}^{p+q}(E, F) = 0$ which shows that $\mathrm{Hom}^q(E, H^p_{\mathcal{A}}F) = 0$ for all p, q as required. □

Note if $\mathcal{D}$ has an exceptional object E that lies in a heart $\mathcal{A}$ such that $\mathbb{S}E \in \mathcal{A}[1]$ under the Serre functor, then the hypotheses of the above theorem are satisfied. Indeed, for $F \in \mathcal{A}$ and $q \geqslant 2$ we have $\mathrm{Hom}(E, F[q]) = \mathrm{Hom}(F[q], \mathbb{S}E) = 0$.

Proposition 3.9. *In the above setting, if Z gives a stability condition on $\mathcal{A}$ with $\mathrm{Im}Z(E) > 0$ then $Z|_{\mathcal{A}\cap E^\perp}$ is a stability condition on $\mathcal{A} \cap E^\perp$.*

Example 3.10. For a del Pezzo surface S with polarisation H, one can consider the corresponding weak stability condition and note that there is an exceptional collection $E_1, \ldots, E_m$ such that for all i, j we have $\mu_H(E_i) > \mu_H(E_j) - H \cdot (-K_S)$. If all E_i are semistable (this holds for the canonical polarisation, for example) then the Kuznetsov component ${E_1}^\perp \cap \ldots \cap {E_m}^\perp$ has a bounded t-structure obtained as follows. Take $\mu \in \mathbb{R}$ such that $\mu_H(E_i) - H(-K_S) \leqslant \mu < \mu_H(E_i)$ for all i, and consider the torsion pair $(\mathcal{C}oh^{>\mu} S, \mathcal{C}oh^{\leqslant\mu} S)$ on $\mathcal{C}oh\, S$. The tilt $\mathcal{C}oh^\mu S$ contains all E_i, and we have $\mathbb{S}E_i = E_i(K_S)[2] \in \mathcal{C}oh^\mu[1]$ by a simple slope computation. Thus $\mathcal{C}oh^\mu S \cap {E_1}^\perp \cap \ldots \cap {E_m}^\perp$ is the heart of a t-structure.

As a corollary, we see that such collections on del Pezzo surfaces have no phantoms. Indeed if $K_0({E_1}^\perp \cap \ldots \cap {E_m}^\perp)$ is torsion, then there are no stability conditions and hence we must have ${E_1}^\perp \cap \ldots \cap {E_m}^\perp = 0$ (otherwise we could restrict stability conditions from $\mathcal{C}oh\, S$.)

The above ideas can be used to show the following.

Theorem 3.11. *For any Fano 3-fold of index $\leqslant 2$, there is a stability condition on the Kuznetsov component.*

It was shown by Pertrusi and Yang that these stability conditions are often invariant under the Serre functor on the Kuznetsov component. Moreover often there is a unique stability condition with such invariance property. This leads to Torelli-type theorems.

Example 3.12 (The cubic 3-fold). Recall that $D(Y_3) = \langle \mathrm{Ku}(Y_3), \mathcal{O}_{Y_3}, \mathcal{O}_{Y_3}(1)\rangle$ so let i be the inclusion of the Kuznetsov component. Consider the pulled back skyscraper $i^*\mathcal{O}_y$ for $y \in Y_3$, which sits in an exact sequence $I_y(1) \hookrightarrow \mathcal{O}_{Y_3}(1) \twoheadrightarrow \mathcal{O}_y$. But there is a surjection $\mathcal{O}_{Y_3}^{\oplus 4} \twoheadrightarrow I_y(1)$ with kernel K_y, and we see that $i^*\mathcal{O}_y = K_y(2)$.

It is then a theorem that the sheaves K_y belong to a four dimensional moduli space of stable sheaves $\mathcal{M}(v)$, where v is the Chern character. In particular there is an embedding $Y_3 \hookrightarrow \mathcal{M}(v)$ given by $y \mapsto K_y$. Sheaves E in the complement $\mathcal{M}(v) \setminus Y_3$ arise from the following construction: pick a cubic surface $S_3 \subset Y_3$. This has seventy two maps to $\mathbb{P}^2$, and for any $\pi : S_3 \to \mathbb{P}^2$ we have that $\pi^*\mathcal{O}(1)$ is a divisor of the form $H + L_1 - L_0$. Here $L_1 - L_0$ is a root of the associated E_6 root system. More importantly, there is a short exact sequence $E \hookrightarrow \mathcal{O}_{Y_3}^{\oplus 3} \twoheadrightarrow \mathcal{O}_{S_3}(H + L_1 - L_0)$ which shows E is locally free. It follows that Y_3 is precisely the locus of non locally-free sheaves in $\mathcal{M}(v)$.

But the moduli space also has an Abel-Jacobi map $\mathcal{M}(v) \to \mathrm{I}\mathcal{J}ac\, Y_3 = \mathrm{CH}_1(Y_3)$ and under this map, we have $K_y \mapsto 0$ while other objects E get mapped to $[L_1] - [L_0]$. In their analysis of the theta divisor $\Theta \subset \mathrm{I}\mathcal{J}ac\, Y_3$, Clemens and Griffiths show that it is given by classes of the form $[L_1] - [L_0]$. Thus $\mathcal{M}(v)$ maps surjectively onto Θ and $Y_3 \subset \mathcal{M}(v)$ lies in the kernel. Computing normal bundle sheaves, they show that $0 \in \Theta$ coincides with $0 \in \mathrm{Cone}(Y_3)$ formally locally. The map is an isomorphism away from $Y_3 \in \mathcal{M}(v)$, $0 \in \Theta$. Thus Θ is smooth away from 0 and Y_3 is the exceptional locus of the blow-up, giving another way to derive the Torelli theorem for cubic 3-folds.

What does this have to do with Kuznetsov components? In this case we can show $\mathcal{M}(v)$ is also the moduli space of all stable objects in $\mathrm{Ku}(Y_3)$. As a corollary, we get a very short proof of the theorem that $\mathrm{Ku}(Y_3)$ determines Y_3. Indeed there is a unique stability condition invariant under the Serre functor, and this gives a four dimensional moduli space $\mathcal{M}(v)$. The 3-fold Y_3 lies in this space as the union of all rational curves.