Homological Mirror Symmetry [EXPLICIT]

(Based on many explanations by Danil, Nick, Luca, Franco.)

Parth Shimpi.

(For the reading group "HMS for Fanos")

8 October, 2024

§ HMS for $\mathbb{P}^1$

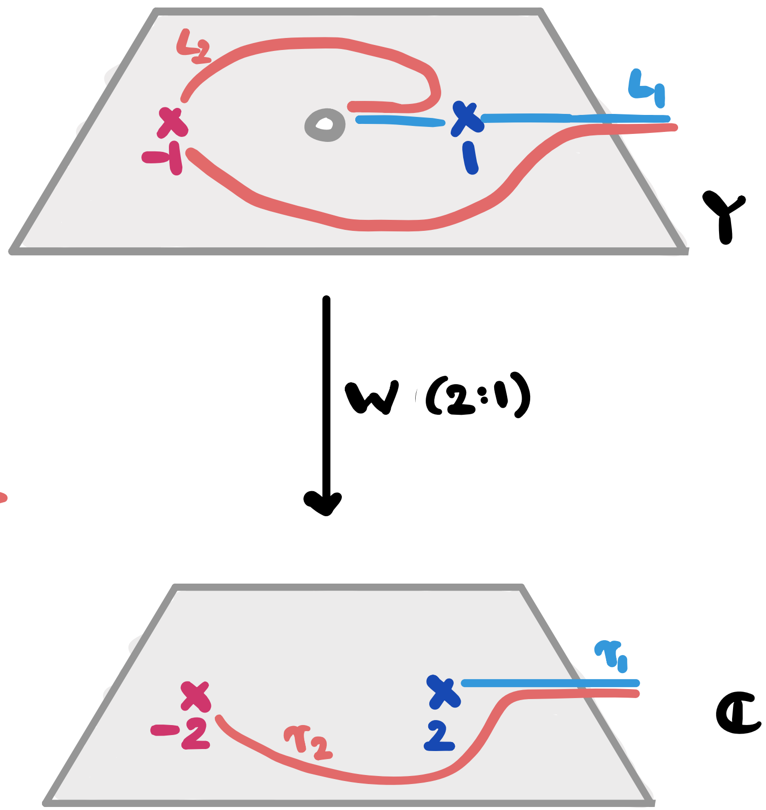
Claim the LG model given by $(Y = \mathbb{C}^*, W = z + z^{-1})$ works.
(symplectic form $dz \wedge d\bar{z} / z\bar{z}$)

This is a Lefschetz fibration

Critical points ± 1, Critical values ± 2

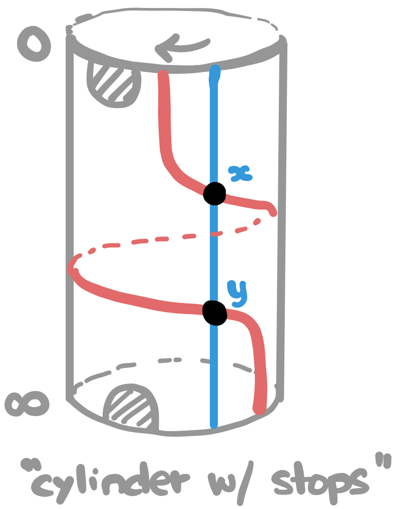
$\Rightarrow$ Get Lagrangians L_1, L_2 coming from lifts of vanishing paths γ_1, γ_2

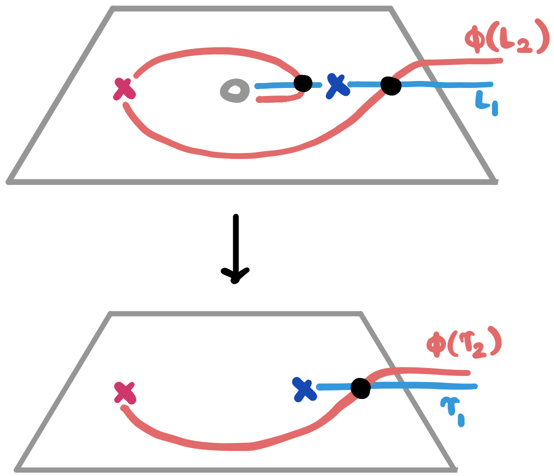
And $FS(Y, W) \cong \langle L_1, L_2 \rangle$



Dimensions and degrees of Hom spaces:

L_1, L_2 exceptional $\Rightarrow$ $\mathrm{Hom}^\bullet(L_i,L_j) \cong \begin{cases} \mathbb{C}\langle e_i\rangle & \text{if } i=j \\ 0 & \text{if } i=1, j=2 \end{cases}$ (deg 0)





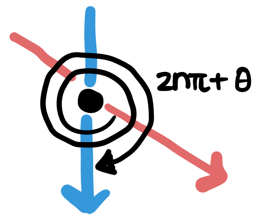
$$\mathrm{Hom}^\bullet(L_1,L_2) \cong \mathbb{C}\langle L_1 \cap \phi(L_2)\rangle$$
$$= \mathbb{C}\langle x, y\rangle$$
(2 dimensional)

What about degrees? The Lagrangians are actually equipped with lifts of phase maps (wrt the volume form $dr + i\,d\theta$ on the cylinder)

L_1 has constant phase ($\downarrow$) so wlog lift to the map $L_1 \to \mathbb{R}$, $\ell \mapsto 0$

$\phi(L_2)$ has phase ($\downarrow$) $0 \leq \theta < \pi/2$ ($\rightarrow$) so lift to $\ell \mapsto 2n\pi + \theta(\ell)$

Then $\deg(\bullet)$ is Maslov index of the path



in Lag.Gr.($\mathbb{R}^2$)

ie $\deg(x) = \deg(y) = n$

Have picture of category
$$\begin{tikzcd} L_1 \arrow[loop below, dashed, "e_1"] \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] & L_2 \arrow[loop below, dashed, "e_2"] \end{tikzcd}$$
, both arrows degree n.

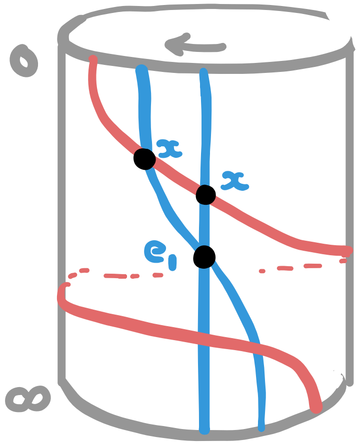
μ^d has degree $2-d$, so degree considerations $\Rightarrow \mu_d = 0$ for $d \neq 2$

$$\left(\text{eg for } \mu_3 : \underbrace{(L_2,L_2)\otimes(L_1,L_2)\otimes(L_1,L_1)}_{\text{each term has degree } n} \to (L_1,L_2) \right)$$

has nothing of degree $n+2-d = n-1$

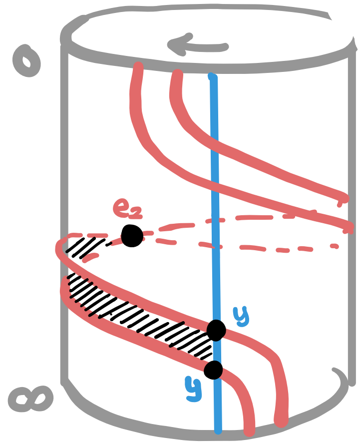
μ^2 can be computed explicitly

$(L_1,L_2)\otimes(L_1,L_1)\longrightarrow(L_1,L_2)$



$$e_1 x = x e_2 = x$$
$$e_2 y = y e_2 = y$$

$(L_2,L_2)\otimes(L_1,L_2)\to(L_1,L_2)$



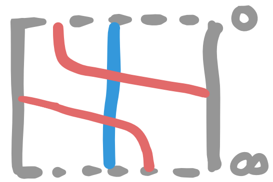
Consequently, $FS(Y,W) \cong \mathrm{End}^{\bullet}_{\mathbb{P}^1}(\mathcal{O}(-1) \oplus \mathcal{O}[n]) \cong \mathcal{A}$

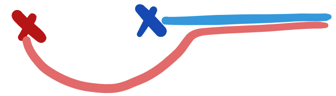
(if $n=0$ then this is the 2-Kronecker algebra with trivial A_∞ structure)

So have homological mirror symmetry — $D^b\mathrm{Coh}\,\mathbb{P}^1$ and $D(FS(Y,W))$ are both equivalent to $D^{\pi}(\mathcal{A}) = \Delta$ closure of $\mathcal{A}$ in $\mathrm{mod}\,\mathcal{A}\,[\mathrm{qisom}^{-1}]$

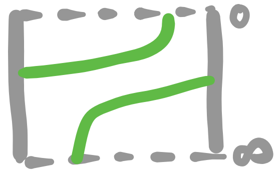
Object in $D^b\mathrm{Coh}\,\mathbb{P}^1$ | Mirror lagrangian / V. path

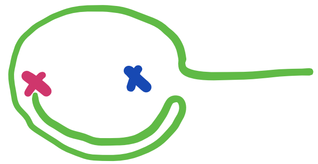
$\mathcal{O}(-1)$, $\mathcal{O}$



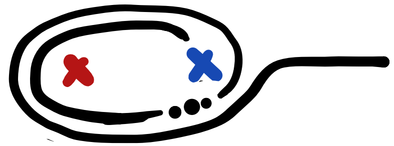


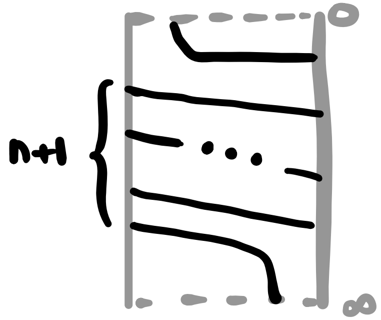
$\mathcal{O}(-2) = \mathbb{L}_{\mathcal{O}(-1)}(\mathcal{O})$





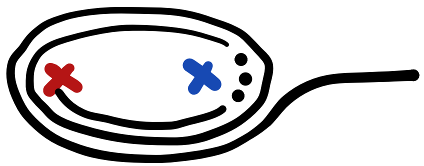
$\mathcal{O}(n)$ for n odd





($\frac{n+1}{2}$ ↻ loops)

$\mathcal{O}(n)$ for n even

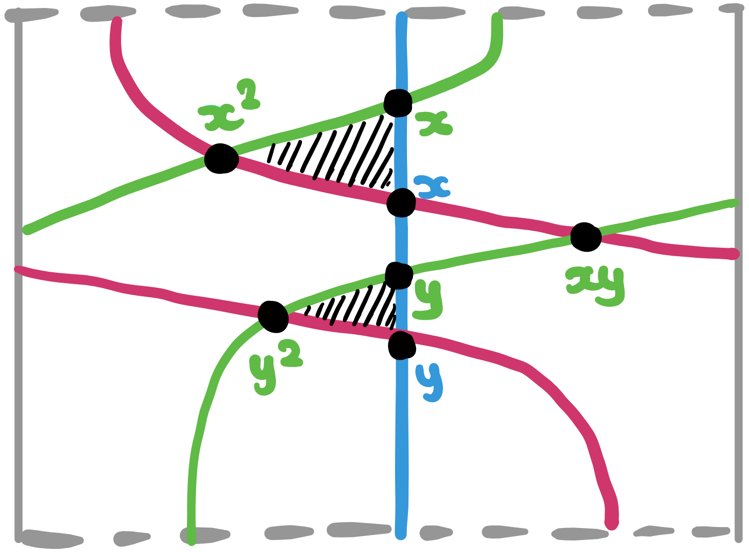


$\mathcal{O}_p$ for $p \neq 0,\infty$ is mirror to

with local system given by $\pi_1(L) \xrightarrow[p]{} GL_1(\mathbb{C})$

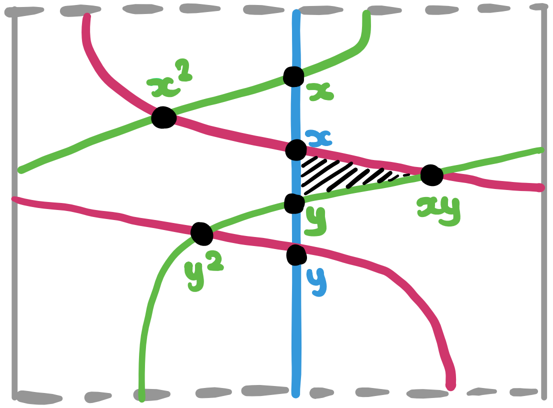
and $-\otimes\mathcal{O}(1)$ is Dehn twist along L (corresponds to taking cone)

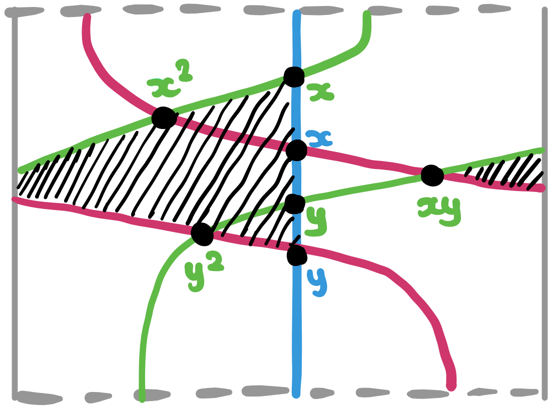
Visualising the compositions $\mathcal{O}(-2) \overset{x}{\underset{y}{\rightrightarrows}} \mathcal{O}(-1) \overset{x}{\underset{y}{\rightrightarrows}} \mathcal{O}$



$x \cdot x = x^2$

$y \cdot y = y^2$





$x \cdot y = xy = y \cdot x$

§ HMS for $\mathbb{P}^2$ and friends

Claim the mirror LG model is $(Y=(\mathbb{C}^*)^2, W=x+y+x^{-1}y^{-1})$

How to arrive at this? At least two ways:

① If X is a toric variety with dense torus $M\otimes\mathbb{C}^*$ and monomial lattice $N=M^\vee$ then $(M\otimes\mathbb{C}^*, N\otimes\mathbb{C}^*)$ is a mirror pair.

Each boundary divisor in M gives a ray in fan of X $(\subseteq M\otimes\mathbb{R})$ and this ray has primitive generator $m\in M$, monomial on $N\otimes\mathbb{C}^*$

Then X has mirror $(Y=N\otimes\mathbb{C}^*, W=\sum_{D\subset\partial X} m_D)$.

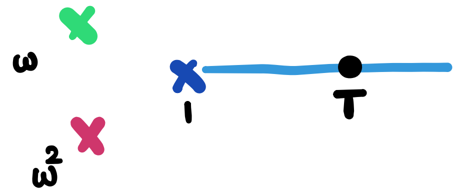
② $X=\mathbb{P}^2$ has anticanonical sections $\sigma_0=xyz$, $\sigma_1=x^2y+y^2x+z^3$ giving a pencil of cubics $E_t=\{t\sigma_0+\sigma_1=0\}$ / $t\in\mathbb{P}^1$

then $\sigma_1/\sigma_0: \underbrace{X\setminus E_\infty}_{\substack{\mathbb{P}^2\setminus\{xyz=0\}\\ =(\mathbb{C}^*)^2}} \longrightarrow \mathbb{C}$ is a Lefschetz fibration.

W has critical points $\{1, \omega, \omega^2\}$ $(\omega = e^{2\pi i/3})$, critical values $\{3, 3\omega, 3\omega^2\}$

Generic fiber $W^{-1}(T)$ is a torus with three punctures

(E_T is a plane cubic meeting E_∞ in $(0:1:0)^4_,\ (1:0:0)^4,\ (1,-1,0)^1$)

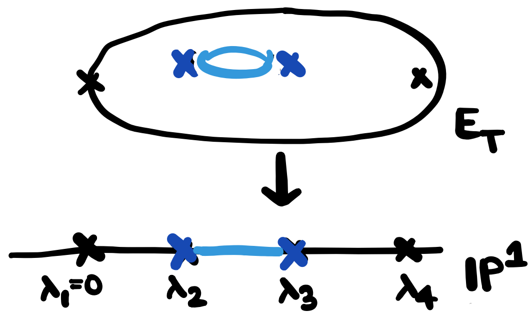
Consider the vanishing path 

. What cycle vanishes?

For $T>1$ real, consider the map $x/z : E_T \to \mathbb{P}^1$

This is a branched double cover,

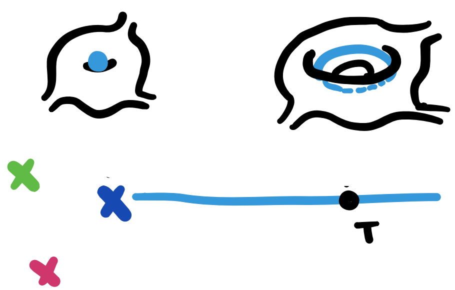
branched at sol^{n}s of $\lambda(\lambda^3+2T\lambda^2+T^2\lambda-4)=0$



As T approaches 1, λ_2 approaches λ_3 along blue path

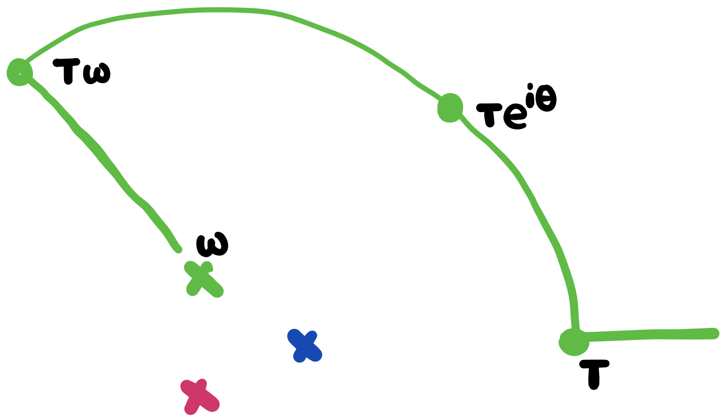
ie the preimage of the blue path shrinks to a point.

$$\begin{array}{c} Y \\ \downarrow W \\ \mathbb{C}^* \end{array}$$

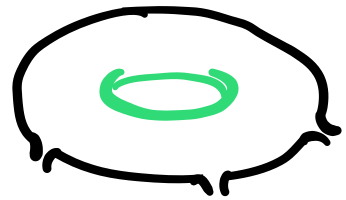


To find the next vanishing cycle in $W^{-1}(T)$,

consider the vanishing path as follows:

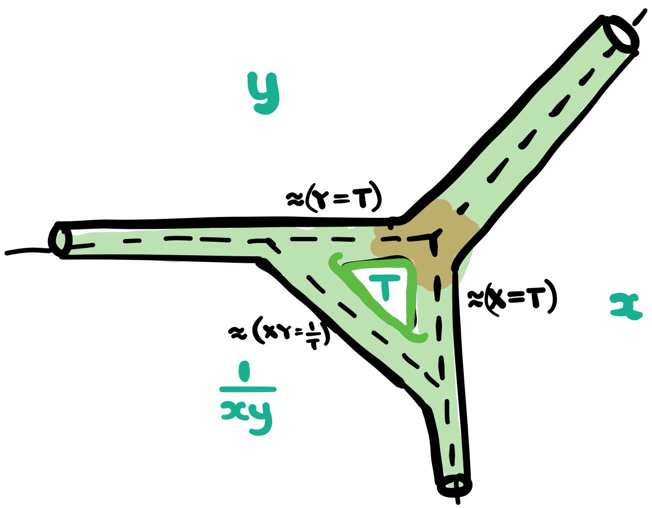


The cycle in $W^{-1}(T\omega)$ is as before,

i.e 

Need to analyse parallel transport along $Te^{i\theta}$.

Idea: For T large, use tropical geometry to find dominant terms.



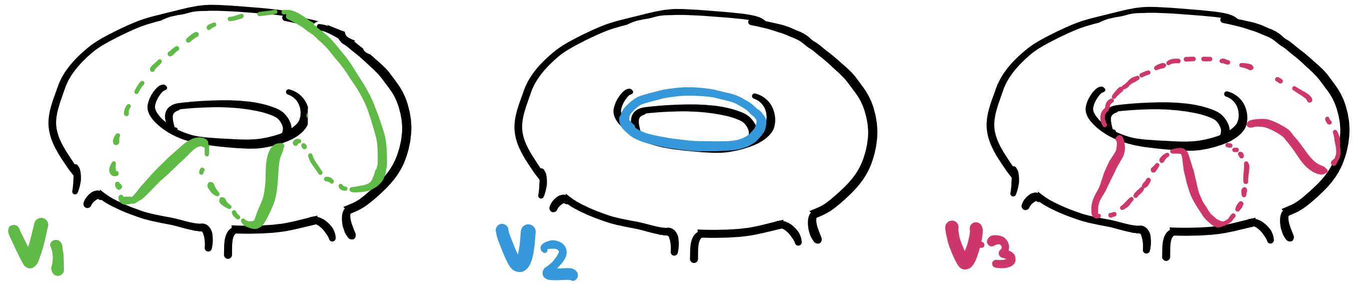
So eg as $Te^{i\theta}$ varies from $\theta = 2\pi/3$ to 0

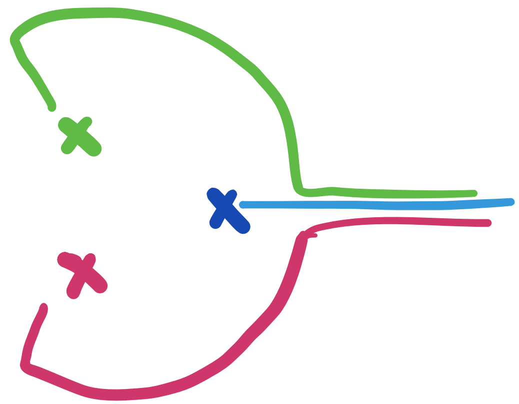
the orange region which looks like $X+Y \approx Te^{i\theta}$ undergoes

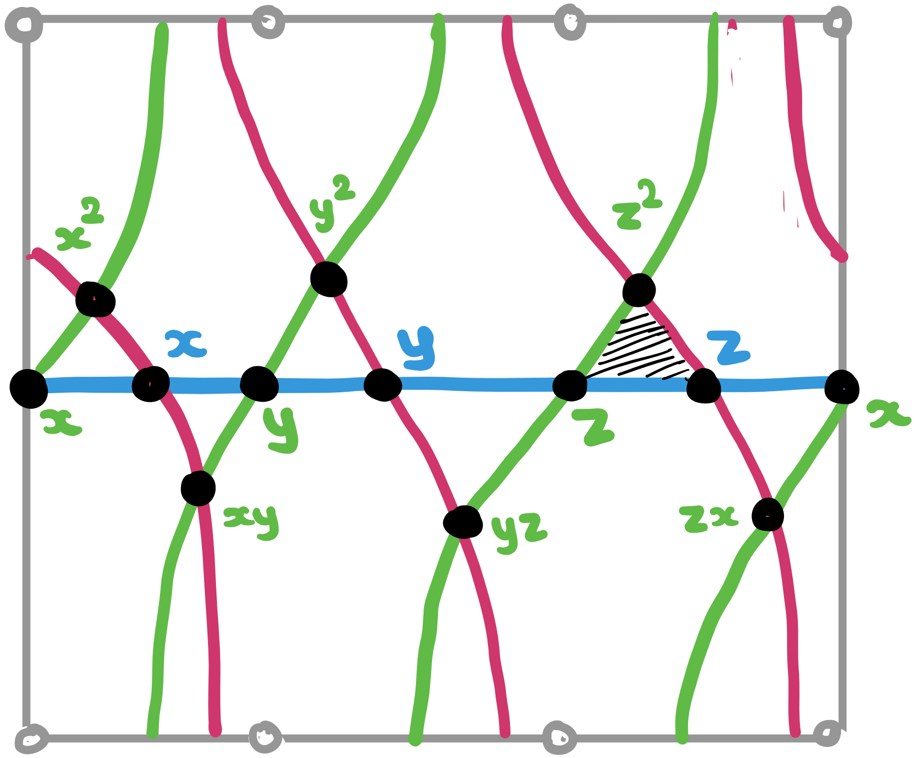
the transformation $(X,Y) \mapsto (Xe^{-i\theta}, Ye^{-i\theta})$.

This tells how the red cycle twists in $W^{-1}(T)$.

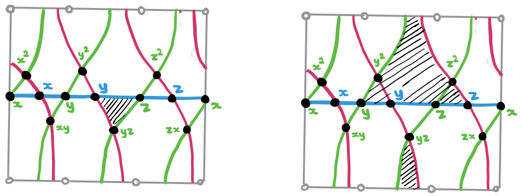
The three vanishing cycles / paths therefore are







$\Big($ $xy = yx = xy$



$\Big)$

For $i<j$, $\mathrm{Hom}^\bullet(L_i, L_j) = \mathrm{Hom}^\bullet(V_i, V_j)$

so have

$$\begin{tikzcd} L_1 \arrow[r, bend left, "x"] \arrow[r, "y"] \arrow[r, bend right, "z"'] & L_2 \arrow[r, bend left, "x"] \arrow[r, "y"] \arrow[r, bend right, "z"'] & L_3 \end{tikzcd}$$

$$\mathrm{Hom}^\bullet(L_1, L_3) = \left\langle \begin{array}{l} x^2 = x \cdot x \\ y^2 = y\,y \\ z^2 = z\,z \\ xy = xy = yx \\ yz = yz = zy \\ zx = zx = xz \end{array} \right\rangle$$

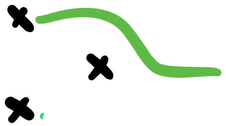
All degree 0 so $\mu_{\geq 3} = 0$

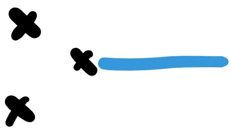
$$FS(Y,W) = \langle L_1, L_2, L_3 \rangle \cong \mathcal{E}nd_{\mathbb{P}^2}\left(\mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O}(1)\right)$$

$$D^{\pi}(FS(Y,W)) \cong D^b\mathcal{C}oh(\mathbb{P}^2).$$

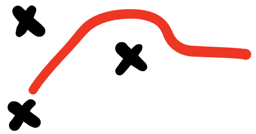
Object in $D^b\mathrm{Coh}\,\mathbb{P}^2$ / Vanishing path / Vanishing cycle

given by class in π_1 of torus

$\Omega^2(2)=\mathcal{O}(-1)$:  : $(1,-3)$

$\mathcal{O}$:  : $(1,0)$

$\mathcal{O}(1)$:  : $(1,3)$

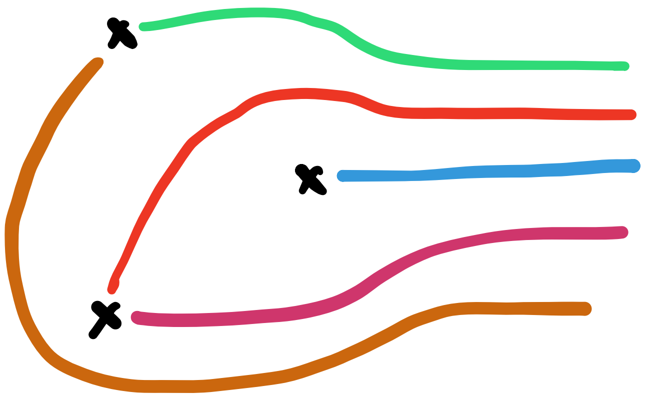
$\Omega^1(1)=\mathbb{L}_{\mathcal{O}}(\mathcal{O}(1))$: 

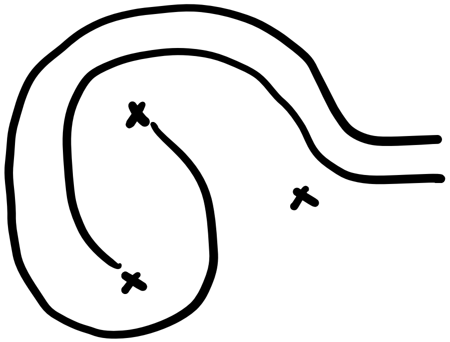
Class of cycle in homology is

$[\mathcal{O}(1)] - (\#\cap)\cdot[\mathcal{O}]$

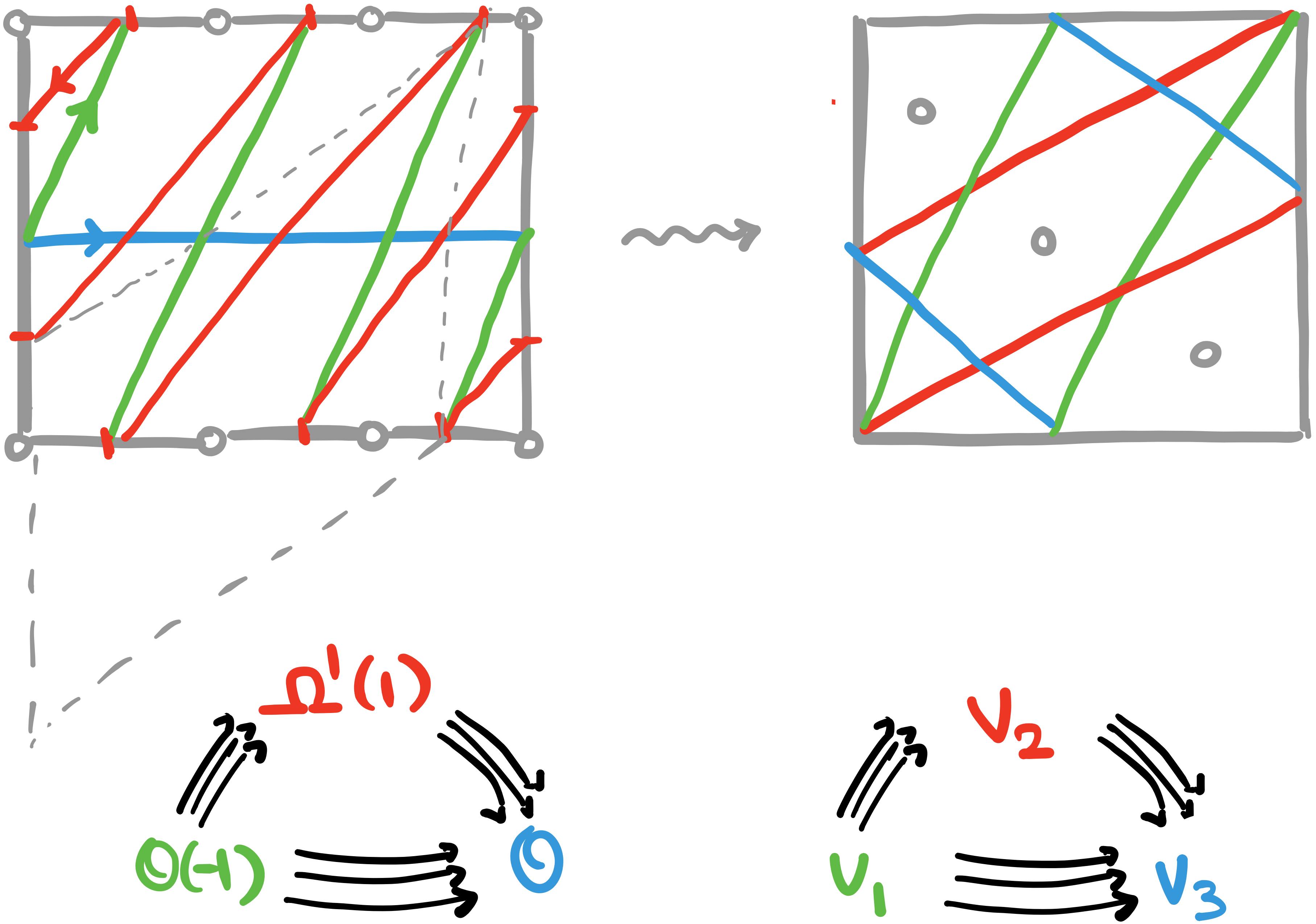
$=(1,3)-3\cdot(1,0)$

$=(-2,3)$

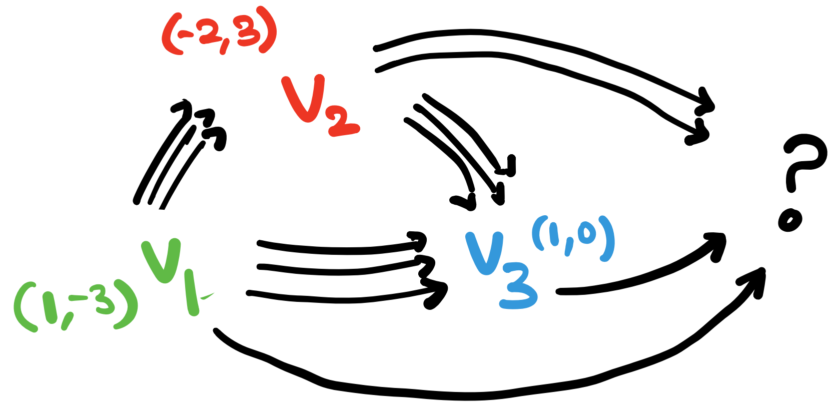




So the exceptional set $\Omega^2(2)$, $\Omega^1(1)$, $\mathcal{O}$ looks like

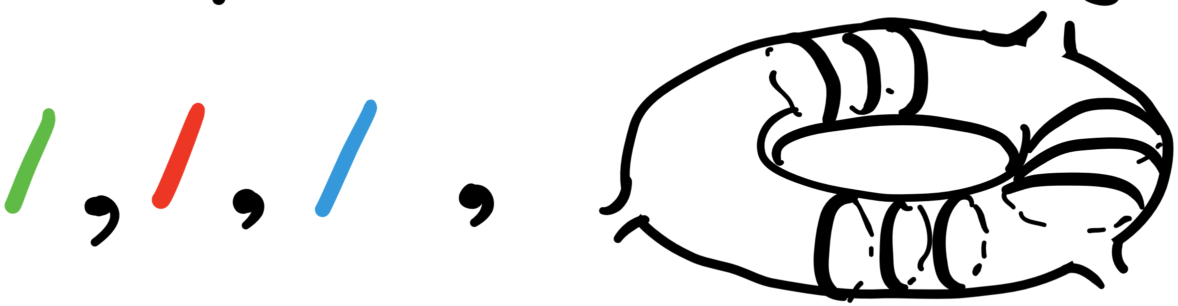


If we wanted to blow up a point on $\mathbb{P}^2$ then the quiver should become



Note the Lagrangian corresponding to (0,1) has the required intersection numbers!

[Aurox-Katzarkov-Orlov] Find an LG model $Y_k \to \mathbb{C}^*$ mirror to $Bl_k\,\mathbb{P}^2$ such that vanishing cycles look like



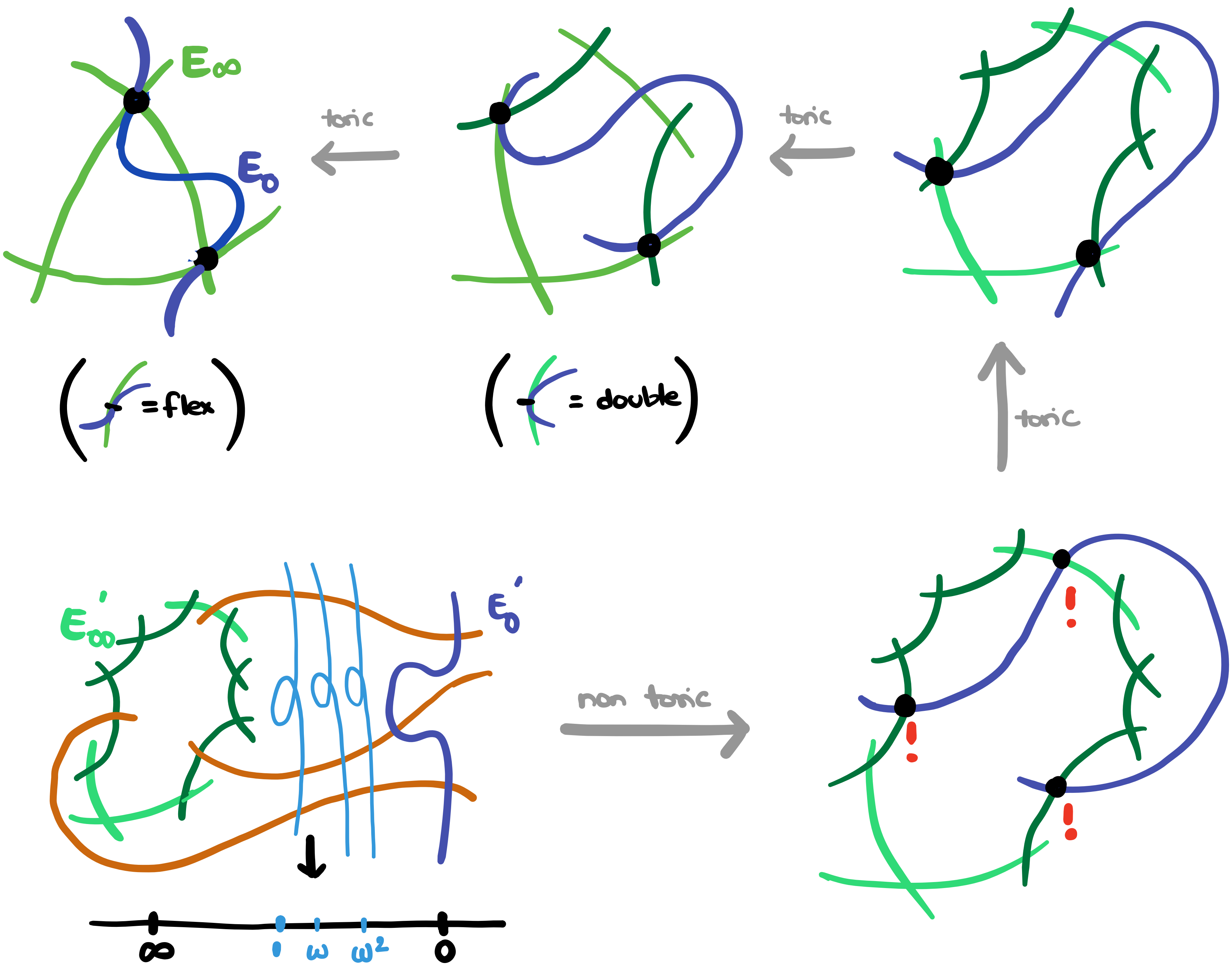
(k rings)

Okay so the LG model for $\mathbb{P}^2$ is iffy
(fibers non-compact)
The "actual" mirror is a rational elliptic surface obtained by compactifying.

$$\begin{tikzcd}
(\mathbb{C}^*)^2 \arrow[r, hook] \arrow[d, "W"] & \mathbb{P}^2 \arrow[d, dashed, "\sigma_1/\sigma_0"] \\
\mathbb{C}^* \arrow[r, hook] & \mathbb{P}^1
\end{tikzcd}$$

The rational map came from pencil E_t in $\mathbb{P}^2$
where $E_0 = \mathbb{V}(\sigma_1)$ smooth cubic
$E_\infty = \mathbb{V}(\sigma_0)$ axes

Indeterminacy where σ_0, σ_1 both vanish, ie $E_0 \cap E_\infty$
Resolve by blowing up.



E∞
E0
toric
toric
= flex
= double
toric
E'∞
E'0
non toric
∞
1
ω
ω²
0

ie mirror to $\mathbb{P}^2$ is a rational elliptic surface w/ an I_9 fiber at ∞, and 3 distinguished sections.

Some technology $\Rightarrow$ Fukaya-Seidel category stays the same if you remove E_∞ and the sections.

To find mirror of $Bl_k\mathbb{P}^2$, [AKO] deform the above potential so that k of the 9 critical points of E_∞ are mapped to something finite instead. The fibration then is