Torsion pairs and 3-fold flops

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Abstract This paper classifies t-structures on the local derived category of a 3-fold flopping contraction, that are intermediate with respect to the heart of perverse coherent sheaves. Equivalently, this describes the complete lattice of torsion classes for the associated modification algebra. The intermediate hearts are (1) categories of coherent sheaves on birational models and tilts thereof in skyscrapers, (2) algebraic t-structures described in the homological minimal model programme, or (3) combinations of the above over appropriate open covers. An analogous classification is also proved for minimal (and partial) resolutions of Kleinian singularities, thus providing a description of all torsion pairs in the module categories of (contracted) affine preprojective algebras. The results have immediate applications to the classification of spherical modules and (semi)bricks, and are first steps towards describing all t-structures and spherical objects in derived categories of surfaces and 3-folds.

It is a truth universally acknowledged, that a mathematician in possession of a triangulated category, must be in want of tools to understand its autoequivalences. This truth guides the homological algebraist's pursuit of t-structures, the representation theorist's pursuit of torsion pairs, and the geometer's of spherical objects.

Such is also the predicament we find ourselves in; this paper studies the derived category of a Gorenstein terminal 3-fold X appearing in a flopping contraction $\pi : X \to Z = \operatorname{Spec}(R, \mathfrak{m})$ over a complete local base. The assumptions on the singularities of X are equivalent to requiring that R is an isolated compound Du Val (cDV) singularity. The bounded derived category of coherent sheaves $\mathbf{D}^{\mathrm{b}}X$, as well as the full subcategory supported on the exceptional fiber

$$\mathbf{D}^0X = \left\{ x \in \mathbf{D}^{\mathrm{b}}X \;\middle|\; \operatorname{Supp} x \subseteq \pi^{-1}[\mathfrak{m}] \right\}$$

has been of interest to birational and symplectic geometers alike [Asp03; Tod08; HW18; HW23; KS24]. The same can be said of the analogous situation in dimension 2, where Z is a canonical surface singularity and X a (partial) resolution [Cra00; IU05; Bri09; IUU10; BDL23]. In either setting, an understanding of autoequivalences, t-structures, and spherical objects is desirable.

When X is smooth, the autoequivalences of the derived category are sometimes controlled by ***spherical objects***. In general they are not, so Hara–Wemyss [HW24] argue for the study of broader collections of objects which behave like the simples of an Abelian category, namely ***(semi)bricks***. The rich interplay between semibricks and t-structures has been a staple tool for representation theorists [MS17; Asa20], and Hara–Wemyss use this to study the null subcategory $\mathcal{C} = \ker(\mathbf{R}\pi_*) \subseteq \mathbf{D}^{\mathrm{b}}X$ which is known to be the 'finite-type' counterpart to $\mathbf{D}^0X$, the latter exhibiting 'affine' behaviour [Bri09; HW23].

Hara–Wemyss show that a global classification of t-structures and bricks in $\mathcal{C}$ is indeed possible, and the classification of spherical objects (when they exist) comes as a corollary. Up to well-understood mutation autoequivalences, the only t-structures are the finitely many hearts $H_1, \ldots, H_n \subset \mathcal{C}$ given by the homological minimal model programme [Wem18], and each is described as the module category of some finite dimensional algebra $\Lambda_{i,\mathrm{con}}$ $(i = 1, \ldots, n)$. Further every brick in $\mathcal{C}$ is the track of some simple $\Lambda_{i,\mathrm{con}}$-module.

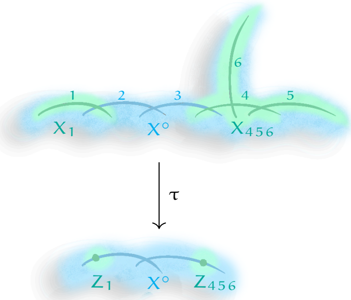
Hearts on the 'affine' category. The category $\mathbf{D}^0X$, too, has algebraic hearts. Van den Bergh [Van04] observed that there is a module-finite R-algebra Λ and a derived equivalence $\mathrm{VdB} : \mathbf{D}^{\mathrm{b}}X \to \mathbf{D}^{\mathrm{b}}\Lambda$ that identifies $\mathcal{fl}\mathrm{mod}\,\Lambda$ with the full subcategory of ***perverse sheaves*** $\operatorname{per}\left(\frac{X}{Z}\right) \subset \mathbf{D}^0X$. This serves as our reference heart. The homological minimal model program extends this to a family of R-algebras $\{{}_\nu\Lambda_\nu \mid \nu \text{ a sequence of mutations}\}$ called ***modification algebras*** that are derived equivalent to Λ. Tracking the natural hearts $\mathcal{fl}\mathrm{mod}\,{}_\nu\Lambda_\nu$ across these equivalences then produces more algebraic t-structures on $\mathbf{D}^0X$ (§§ 3.4 and 3.5) which we say are ***mutations of*** $\operatorname{per}\left(\frac{X}{Z}\right)$.

However it is evident that there can be no autoequivalence which identifies the above algebraic t-structures with the geometric (and non-Artinian) heart $\operatorname{coh} X$. Other geometric hearts can be constructed by iteratively flopping exceptional curves in X to obtain new flopping contractions $\pi : W \to Z$, we say the 3-fold W thus obtained is a *birational model* of X. Indeed flops give derived equivalences [Bri02; Che02], so tracking $\operatorname{coh} W$ across the composite Bridgeland–Chen functor $\Psi : \mathbf{D}^b W \to \mathbf{D}^b X$ gives a geometric t-structure on $\mathbf{D}^0 X$.

To make matters worse, there are hearts which are 'algebro-geometric' in quite the literal sense. The construction of the perverse heart is local, so any crepant morphism $\tau : X \to Y$ contracting some (but not all) of the π-exceptional curves gives a sheaf $\mathscr{A}$ of coherent $\mathcal{O}_Y$-algebras and a derived equivalence $\mathbf{D}^b X \to \mathbf{D}^b \mathscr{A}$. This identifies a category of suitable $\mathscr{A}$-modules with a heart $\operatorname{per}(\frac{X}{Y}) \subset \mathbf{D}^0 X$ whose objects are precisely complexes that look like finite length Λ-modules on the τ-exceptional locus and like $\mathcal{O}_X$-modules elsewhere (theorem 5.23). The category is again not equivalent to any heart examined previously, and it is hence apparent that any classification must also account for these (semi-)geometric hearts.

Classification of intermediate hearts. The main result of this paper shows that, at least for sufficiently small cohomological spread, the above possibilities are in fact exhaustive.

This involves building semi-geometric hearts locally using algebraic and geometric categories for 'smaller' flopping contractions (§ 5.4), thus obtaining an inductive description of t-structures. Indeed any partial contraction $\tau : X \to Y$ is an isomorphism away from the open locus X° of its non-exceptional fibers, while in any sufficiently small neighbourhood of a τ-exceptional fiber C_I, the map τ restricts to a flopping contraction $X_I \to Z_I$ with complete local base Z_I in Y. This gives a (flat) cover of X. We show any intermediate t-structure (i.e. a t-structure whose heart is concentrated in cohomological degrees 0 and -1) decomposes into purely algebraic or purely geometric hearts with respect to some such cover.



Theorem A (= C, 5.27). *Let K be the heart of a t-structure on $\mathbf{D}^0 X$ that is contained in $\operatorname{per}(\frac{X}{Z})[-1, 0]$. Then there is a birational model W of X, and a partial contraction $\tau : W \to Y$, such that K satisfies the following after being pulled back across the Bridgeland–Chen equivalence $\Psi : \mathbf{D}^0 W \to \mathbf{D}^0 X$.*

(1) On the locus $W^\circ \subset W$ where τ is an isomorphism, $\Psi^{-1} K$ restricts to the category of coherent sheaves $\operatorname{coh} W^\circ$, possibly tilted in skyscraper sheaves $\langle \mathcal{O}_p \mid p \in Q \rangle$ for some subset of closed points $Q \subset W^\circ$.

(2) For each τ-exceptional fiber C_I in W, for sufficiently small neighbourhoods $W_I \supset C_I$ and $Z_I \ni \tau(C_I)$, the restriction of $\Psi^{-1} K$ to a W_I is an (algebraic) mutation of the category $\operatorname{per}\left(\frac{W_I}{Z_I}\right) \subset \mathbf{D}^0 W_I$.

Moreover, the above data uniquely determines K.

In particular if τ above contracts all exceptional curves, then K is an algebraic mutation of $\operatorname{per}(\frac{W}{Z})$ (hence also of $\operatorname{per}(\frac{X}{Z})$) and thus K is the category of finite length modules over a modification algebra. On the other extreme if τ contracts no curves then K is a geometric t-structure on the birational model W.

Classification of bricks. As a corollary we get a succinct description of bricks in $\operatorname{per}(\frac{X}{Z})$, i.e. π-perverse sheaves whose endomorphism algebra is one-dimensional. This extends to modification algebras a result of Crawley-Boevey [Cra00, lemma 1], who showed that the dimension vector of any brick-module over an affine preprojective algebra is a root.

Theorem B (= 6.13). *Let $b \in \operatorname{per}(\frac{X}{Z})$ be a brick. Then b is either*

(1) a simple module over some modification algebra ${}_\nu\Lambda_\nu$, or

(2) a skyscraper sheaf on some birational model W of X,

tracked under appropriate equivalences induced by mutations or flops. In particular, the $\mathbf{K}$-theory class of b is a primitive restricted root of the (affine) Dynkin data associated to X.

In sufficiently restricted settings (e.g. under the assumptions imposed in [Tod08]), every algebraic mutation of $\mathcal{per}(\frac{X}{Z})$ is equivalent to the category of perverse sheaves on some birational model W. Then the above result says that, up to applying mutation functors, every brick in $\mathcal{per}(\frac{X}{Z})$ is either a point sheaf $\mathcal{O}_p$, or the twisted structure sheaf $\mathcal{O}_{C_i}(-1)$ on some integral curve, or the suspended canonical sheaf $\omega_{\underline{C}}[1]$ of the scheme-theoretic exceptional fiber on some birational model W. In general however there are modifying algebras not related to birational geometry in any obvious way [these are the 'hidden' t-structures alluded to in HW23], and expressing their simple modules in geometric terms is difficult. For a single flopping curve this is accomplished by Donovan–Wemyss [DW24, §4] who show that each such simple is, up to mutation, determined by the structure sheaf or the canonical sheaf of some thickening of the exceptional curve.

Canonical surfaces and preprojective algebras. Suppose $\overline{X}$ is a Gorenstein canonical surface that admits a crepant birational morphism $\pi : \overline{X} \to \overline{Z} = \operatorname{Spec}(\overline{R}, \mathfrak{p})$ with complete local base. The results of this paper, though developed in the context of 3-folds, apply verbatim to the derived category of $\overline{X}$ and its π-perverse heart provided one suitably reinterprets the notions of modification algebras and birational models.

Such a map π is necessarily a crepant (partial) resolution of the Kleinian singularity $\overline{Z}$, i.e. $\overline{X}$ is obtained by contracting some exceptional curves in the minimal resolution $\widetilde{X} \to \overline{Z}$. It is well-known that $\widetilde{X}$ is derived equivalent to an affine preprojective algebra Π; the contraction $\widetilde{X} \to \overline{X}$ determines an idempotent $e \in \Pi$ such that $\overline{X}$ is derived equivalent to the *contracted preprojective algebra* $e\Pi e$ [KIWY15]. This equivalence $\mathbf{D}^{\mathrm{fl}}(e\Pi e) \to \mathbf{D}^0(\overline{X})$ is also recovered by Van den Bergh's construction [Van04], and algebraic mutations of the category of π-perverse sheaves can be read off from the tilting theory of $e\Pi e$ [IW, §7.4].

The discussion of birational models is only slightly more subtle, as curves in surfaces do not flop. Birational transformations (including flops in dimension 3), on the other hand, are naturally induced via geometric invariant theory. Indeed $\widetilde{X}$ appears as a moduli space $\mathcal{M}(\theta, \delta)$ of stable Π-modules, for some generic stability parameter θ relative to a $\mathbf{K}$-theory class δ [CS98]. Variation of GIT parameter $\theta_1 \rightsquigarrow \theta_2$ produces a birational map $\mathcal{M}(\theta_1, \delta) \dashrightarrow \mathcal{M}(\theta_2, \delta)$ of $\overline{Z}$-schemes. In particular if θ_1 and θ_2 lie in adjacent chambers then this birational map is defined away from a single exceptional curve, and simple wall–crossing of the GIT parameter thus gives an analogue of flops in dimension 2. The role of Bridgeland–Chen equivalences is played by *reflection functors* [SY13], and the results are extended to the contracted setting by Iyama–Wemyss [IW].

With this, all arguments in §§ 3 to 6 regarding the partial order of algebraic intermediate hearts and its interactions with convex geometry and line bundles on birational models hold verbatim. Thus the reader indifferent to dimensions can safely read the remainder of this paper as if it were written for surfaces.

To the reader interested in multiple dimensions, we remark that the correspondence between our results for 3-folds and surfaces is natural. Indeed, the morphism $\overline{X} \to \overline{Z}$ can be embedded as a generic hyperplane section (*general elephant*) of some 3-fold flopping contraction $X \to \operatorname{Spec}(R, \mathfrak{m})$, and the corresponding non-commutative algebras are related by the reduction $e\Pi e = \Lambda \otimes_R \overline{R}$. By reducing both to the fiber $R/\mathfrak{m} = \overline{R}/\mathfrak{p}$, Kimura [Kim24, theorem 5.4] shows that the corresponding functor $\mathcal{fl}\mathcal{mod}(e\Pi e) \hookrightarrow \mathcal{fl}\mathcal{mod}\Lambda$ induces a bijection of torsion classes. Likewise lemma 5.1 *ibid.* gives the correspondence between bricks.

Mutating v/s tilting. To motivate why a result such as theorem A is desirable towards a full classification, we briefly sketch the key argument of [HW24]. Each algebra $\Lambda_{i,\mathrm{con}}$ produced by the homological minimal model programme is silting discrete [Aug20], thus can be assigned a finite and complete hyperplane arrangement, the *silting fan*. The chambers $\sigma_1, \ldots, \sigma_n$ of this arrangement are in bijection with the hearts $\mathcal{H}_i \subset \mathcal{C}$, and each minimal sequence of wall–crossings $\sigma_j \rightsquigarrow \sigma_i$ is assigned an atomic mutation functor Φ_{ij}. Then, say, given an object $x \in \mathcal{H}_i[-n, 0]$ with non-zero cohomologies in degrees 0 and $n (> 0)$, Hara–Wemyss consider the set of paths $\sigma_j \rightsquigarrow \sigma_i$ such that $\Phi_{ij}(x)$ lies in $\mathcal{H}_j[-n, 0]$ and show that for the longest such path $\sigma_{j_0} \rightsquigarrow \sigma_i$, the object $\Phi_{ij_0}(x)$ in fact lies in $\mathcal{H}_{j_0}[-n+1, 0]$. Iterating shows x lies in some $\mathcal{H}_j$ after a finite sequence of mutations.

Finiteness of the silting fan underpins the argument, guaranteeing that any poset considered has maximal elements. The affine curse ensures we don't have such privileges when studying $\mathbf{D}^0X$, and the silting fan of Λ is infinite and incomplete. So while there still is an assignment of algebraic hearts and mutation functors to chambers and wall-crossings (§ 4.7), the fan has infinite paths which culminate outside its support.

We thus replace mutation and the silting fan with *tilting in torsion pairs* and the *heart fan* respectively. Given any Abelian category H, Happel–Reiten–Smalø [HRS96] show that the set $\operatorname{tilt}(\mathsf{H})$ of intermediate t-structures (i.e. t-structures with hearts contained in $\mathsf{H}[-1,0]$) can be recovered by tilting H in its torsion subcategories, and the containment order of torsion classes enhances $\operatorname{tilt}(\mathsf{H})$ to a *complete lattice* (i.e. a poset in which every subset has an infimum and a supremum). Broomhead–Pauksztello–Ploog–Woolf [BPPW24] assign each heart $\mathsf{K} \in \operatorname{tilt}(\mathsf{H})$ to a cone $\mathbf{C}\,\mathsf{K}$ and show that the ensemble $\operatorname{HFan}(\mathsf{H})$ of heart cones is a fan. This heart fan is complete and simplicial for categories such as $\mathcal{P}er\left(\frac{X}{Z}\right)$, and the silting fan then sits inside $\operatorname{HFan}(\mathsf{H})$ as a sub-fan. Relating atomic mutations to tilting in *functorially finite* torsion classes, we thus have a 'completion' of silting theory.

Giving a complete description of the lattice of torsion classes is hard, even for finite dimensional algebras [Tho21; DIRRT23]. The heart fan gives some insight into tilts satisfying numerical criteria (§ 2.4), but there are examples of tilts of algebraic hearts which have trivial (i.e. $\mathbf{0}$) heart cone [see BPPW24, example 6.7]. In such situations the numerical criteria lead to tautologies, and we are left to our own devices.

The main result of this paper, then, is that such situations do not arise when studying tilts of $\mathcal{P}er\left(\frac{X}{Z}\right)$. Consequently a complete description of the lattice of torsion classes is possible.

Theorem C (= 5.18). *The heart fan of* $\mathsf{H} = \mathcal{P}er\left(\frac{X}{Z}\right)$ *is given by an intersection arrangement associated to the restriction of an affine Dynkin root system, i.e. one of the arrangements described in* [IW]. *The heart cones are described as follows.*

(1) (= 6.6) *The trivial cone* $\mathbf{0}$ *is* not *the heart cone of any intermediate heart.*

(2) (= 4.25, 4.23) *A cone outside the imaginary-root hyperplane* $\{\delta = 0\}$ *is a heart cone if and only if it is full-dimensional, and in this case it is the heart cone of a unique algebraic heart in* $\operatorname{tilt}(\mathsf{H})$. *The positive and negative orthants* $\mathsf{C}^+, \mathsf{C}^-$ *are the heart cones of* H *and* $\mathsf{H}[-1]$ *respectively, and the heart associated to any other full dimensional cone can be expressed as a mutation of* $\mathcal{P}er\left(\frac{X}{Z}\right)$ *by choosing wall–crossing paths from* $\mathsf{C}^{\pm}$.

The hearts H *and* $\mathsf{H}[-1]$ *are the maximal and minimal elements in* $\operatorname{tilt}(\mathsf{H})$ *respectively, and the partial order on remaining algebraic hearts respects (atomic) wall–crossing distance from these.*

(3) (= 5.5, 5.13) *The induced finite hyperplane arrangement on* $\{\delta = 0\}$ *is naturally identified with the movable fan of* X. *Every maximal cone* σ *in this subfan thus parametrises nef divisors on a unique birational model* W *of* X, *and such* σ *is then the heart cone of* $\mathcal{C}oh\,W$ *tracked under the composite Bridgeland–Chen functor* $\mathbf{D}^0 W \to \mathbf{D}^0 X$.

(4) (= 5.28, 5.27) *More generally, every non–trivial cone* $\sigma \subset \{\delta = 0\}$ *can be assigned a unique partial contraction* $W \to Y$ *such that* $\mathcal{P}er\left(\frac{W}{Y}\right)$ *(tracked under flop equivalences) is the maximal heart in* $\operatorname{tilt}(\mathsf{H})$ *with heart cone* σ. *Every other heart with heart cone* σ *can be obtained by arbitrarily mutating the algebraic components of* $\mathcal{P}er\left(\frac{X}{Y}\right)$ *and tilting in skyscrapers in the geometric components.*

An algebraic intermediate heart K *then satisfies* $\mathsf{K} > \mathcal{P}er\left(\frac{W}{Y}\right)$ *if and only if there is an atomic path* $\mathsf{C}^+ \rightsquigarrow \mathbf{C}\,\mathsf{K}$ *which can be atomically extended to an infinite sequence of wall–crossings* $\mathsf{C}^+ \rightsquigarrow \mathbf{C}\,\mathsf{K} \rightsquigarrow \mathbf{C}\,\mathsf{K}_1 \rightsquigarrow \mathbf{C}\,\mathsf{K}_2 \rightsquigarrow \cdots$ *with generic limit in* σ. *In this case* $\mathcal{P}er\left(\frac{X}{Y}\right)$ *is the infimum of the decreasing chain* $\mathsf{K} > \mathsf{K}_1 > \mathsf{K}_2 > \ldots$ *in* $\operatorname{tilt}(\mathsf{H})$.

In §§ 3, 4 and 5 we establish the numerical story, enumerating for each cone in the intersection arrangement all the intermediate hearts and King–semistable objects associated to it. This involves establishing the fact that coherent sheaves on any birational model W (which are a priori only intermediate with respect to the perverse heart on W) are in fact intermediate with respect to the perverse heart on X. For this a careful analysis of the partial order is necessary, and convex geometric tools are needed not only to supply a systematic enumeration scheme but also to establish crucial limiting results as in theorem C (4).

Once the heart fan is filled in, the question of whether there are any hearts hidden away in the $\mathbf{0}$–cone remains. Here the affine blessing ensures that the non-algebraic locus in the heart fan is codimension 1, so every heart in $\operatorname{tilt}(\mathsf{H})$ has tight algebraic bounds. The following example is illustrative.

Single-curve flops. Let $R = \mathbb{C}[\![u, v, x, y]\!]/(uv-xy)$ be the cA_1 singularity, i.e. the base of the Atiyah flop $X \dashrightarrow W$ where X is the neighbourhood of a $(-1,-1)$ rational curve C. The two flopping contractions $X, W \to Z = \operatorname{Spec} R$ are obtained by blowing up the ideals (u, x) and (u, y) respectively.

Accordingly the modification algebra associated to X is $\Lambda = \mathrm{End}(R \oplus (u,x))$, it can be shown that all modification algebras are isomorphic and are related by derived equivalences as below.

$$\begin{array}{ccccccccc}
\cdots \underset{\Psi_1}{\overset{\Psi_1}{\rightleftarrows}} & \mathbf{D}^{\mathrm{fl}}\mathrm{End}\begin{pmatrix} L \\ \oplus \\ (u,y)\end{pmatrix} & \underset{\Psi_0}{\overset{\Psi_0}{\rightleftarrows}} & \mathbf{D}^{\mathrm{fl}}\mathrm{End}\begin{pmatrix} R \\ \oplus \\ (u,y)\end{pmatrix} & \underset{\Psi_1}{\overset{\Psi_1}{\rightleftarrows}} & \mathbf{D}^{\mathrm{fl}}\mathrm{End}\begin{pmatrix} R \\ \oplus \\ (u,x)\end{pmatrix} & \underset{\Psi_0}{\overset{\Psi_0}{\rightleftarrows}} & \mathbf{D}^{\mathrm{fl}}\mathrm{End}\begin{pmatrix} K \\ \oplus \\ (u,x)\end{pmatrix} & \underset{\Psi_1}{\overset{\Psi_1}{\rightleftarrows}} \cdots \\
 & & & \uparrow{\scriptstyle \mathrm{VdB}} & & \uparrow{\scriptstyle \mathrm{VdB}} & & & \\
 & & & \mathbf{D}^0 W & & \mathbf{D}^0 X & & &
\end{array}$$

Here the functors $\Psi_i : \mathbf{D}^{\mathrm{fl}}(\mathrm{End}\, N) \to \mathbf{D}^{\mathrm{fl}}(\mathrm{End}\, M)$ are given as $\mathbf{R}\mathrm{Hom}(\mathrm{Hom}(N,M), -)$, and the index records which summand changes between N and M. This changed summand is computed via *mutation* (theorem 3.7), thus for instance (u,x) and (u,y) are (first) syzygies of each other while $L \subset (u,y)^{\oplus 2}$ is the kernel of the natural map $((u,y) \hookrightarrow R) \oplus ((u,y) \twoheadrightarrow (v,x) \hookrightarrow R)$.

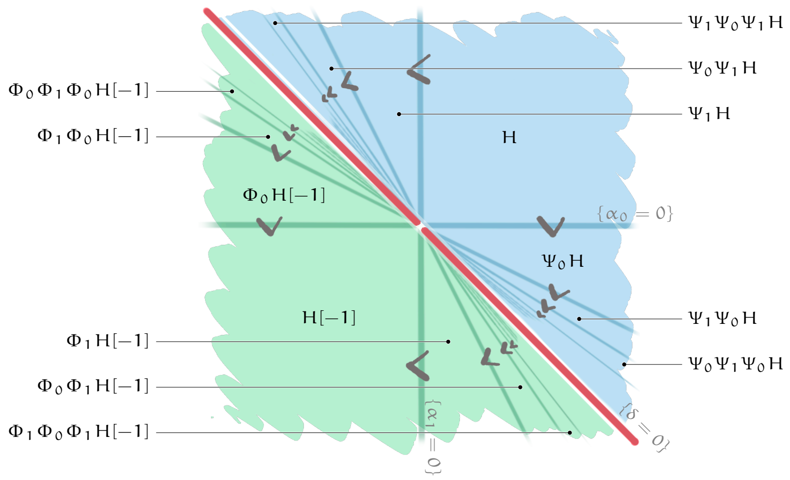
Any category $\mathbf{D}^{\mathrm{fl}}(\mathrm{End}\, M)$ has a natural heart $H_M = \mathcal{flmod}(\mathrm{End}\, M)$, and tracking this across equivalences appearing in the above diagram (and their inverses) produces new t-structures in $\mathbf{D}^0 X$. Such a t-structure is intermediate with respect to the perverse heart when the chain of functors $\mathbf{D}^{\mathrm{fl}}(\mathrm{End}\, M) \rightsquigarrow \mathbf{D}^0 X$ is chosen as short as possible, i.e. the tracks of H_M that lie in $\mathcal{per}(\frac{X}{Z})[-1,0]$ are of the form

$$\mathrm{VdB}^{-1} \circ (\ldots \circ \Psi_0 \circ \Psi_1 \circ \Psi_0 \circ \ldots) H_M \quad \text{or} \quad \mathrm{VdB}^{-1} \circ (\ldots \circ \Psi_0 \circ \Psi_1 \circ \Psi_0 \circ \ldots)^{-1} H_M[-1].$$

Note the specified chain completely determines the domain of the functor, so we may drop the subscript for brevity. We also drop VdB from the notation, so for instance $\Psi_1 H$ is the image of $\mathcal{flmod}(R \oplus (u,y))$ under the functor $\mathrm{VdB}^{-1} \circ \Psi_1$. The algebraic hearts in $\mathbf{D}^0 X$ thus obtained are in natural bijection with full-dimensional cones in the $\widetilde{A}_1$ (∘═•) intersection arrangement, see fig. 1.

Figure 1. The heart fan for a minimal resolution of the cA_1 singularity.

Hyperplanes are induced by the $\widetilde{A}_1$ root system, where the simple real roots α_0, α_1 are identified with the $\mathbf{K}$-theory classes of $\omega_C[1]$ and $\mathcal{O}_C(-1)$ respectively.



The ray $\{\delta = 0, \alpha_0 \geqslant 0\}$, geometrically the 'limit' of the path $\mathbf{C}(H) \to \mathbf{C}(\Psi_0 H) \to \mathbf{C}(\Psi_1\Psi_0 H) \to \ldots$, is the heart cone of $\mathcal{coh}\, X = \inf\{H, \Psi_0 H, \Psi_1\Psi_0 H, \ldots\}$. Likewise it is also the heart cone of the *reversed geometric heart* $\overline{\mathcal{coh}}\, X = \sup\{H[-1], \Phi_1 H[-1], \Phi_0\Phi_1 H[-1], \ldots\}$ which can be obtained by tilting $\mathcal{coh}\, X$ in the torsion class generated by all skyscrapers.

The ray $\{\delta = 0, \alpha_0 \leqslant 0\}$ is similarly seen to be the heart cone of $\mathrm{flop}(\mathcal{coh}\, W)$ and other geometric hearts that live on W, where $\mathrm{flop} : \mathbf{D}^0 W \to \mathbf{D}^0 X$ is the Bridgeland–Chen flop functor (in this case isomorphic to $\mathrm{VdB}^{-1} \circ \Psi_1 \circ \mathrm{VdB}$).

Now the partial order on algebraic hearts (indicated in fig. 1) is such that wall–crossings correspond to simple tilts (§ 4.2), thus $\Psi_0 H$ is obtained by tilting H in the torsion class $\langle \omega_C[1] \rangle$ while $\Phi_1 H[-1]$ and $H[-1]$ are related by a tilt in $\langle \mathcal{O}_C(-1) \rangle$. Considering how simples sit in relation to torsion pairs then allows us to 'push' non-algebraic intermediate hearts $K \in \mathrm{tilt}(H)$ towards the geometric hyperplane.

Algorithm 1.1 (Simple–tilting). Write $\mathsf{H} = \mathsf{T} * \mathsf{F}$ for the torsion pair corresponding to $\mathsf{K}(= \mathsf{F} * \mathsf{T}[-1])$.

1. Since $\mathsf{K} \neq \mathsf{H}$, one of the simples of H (say $S_0 = \omega_C[1]$) lies in T and this forces the inequality $\mathsf{K} < \Psi_0\mathsf{H}$.
2. The inequality $\Psi_0\mathsf{H} > \mathsf{K}$ shows that T intersects $\Psi_0\mathsf{H}$ nontrivially, hence must contain some simple $\{\Psi_0 S_0, \Psi_0 S_1\}$ of $\Psi_0\mathsf{H}$. Since $\Psi_0 S_0 = \omega_C$ lies outside H, we must have $\Psi_0 S_1 \in \mathsf{T}$ and thus $\Psi_1\Psi_0\mathsf{H} > \mathsf{K}$. Iterating, we have $\mathsf{K} < \ldots\Psi_0\Psi_1\Psi_0\mathsf{H}$ for arbitrarily long chains, and hence $\mathsf{K} \leqslant \mathcal{coh}\,X$.
3. On the other hand $\mathsf{K} \neq \mathsf{H}[-1]$ so F contains some simple of H, which in this case must be $S_1 = \mathcal{O}_C(-1)$. This shows $\mathsf{K} > \Phi_1\mathsf{H}[-1]$, and iterating as above gives $\mathsf{K} \geqslant \overline{\mathcal{coh}}\,X$.
4. Since $\overline{\mathcal{coh}}\,X$ and $\mathcal{coh}\,X$ both share the heart cone $\{\delta = 0, \alpha_0 \geqslant 0\}$, K must do so too i.e. K is numerical. In fact K can be expressed as a tilt of $\mathcal{coh}\,X$ in $\langle \mathcal{O}_p \mid p \in Q \rangle$ for some $Q \subset C$.

Thus every heart in $\operatorname{tilt}(\mathcal{per}(\frac{X}{Z}))$ is either algebraic, or can be shown to be geometric via the above recipe.

Multi-curve flops. After accounting for perverse hearts for partial contractions, the above description of the heart fan and the partial order carries over to higher rank cases. Thus for a crepant resolution $X \to Z$ of the cA_2 singularity $Z = \operatorname{Spec} \mathbb{C}[\![u, v, x, y]\!]/(uv - xy(x+y))$, the heart fan of $\mathcal{per}(\frac{X}{Z})$ is described by an $\widetilde{A}_2$ (⁂) root system as in fig. 2.

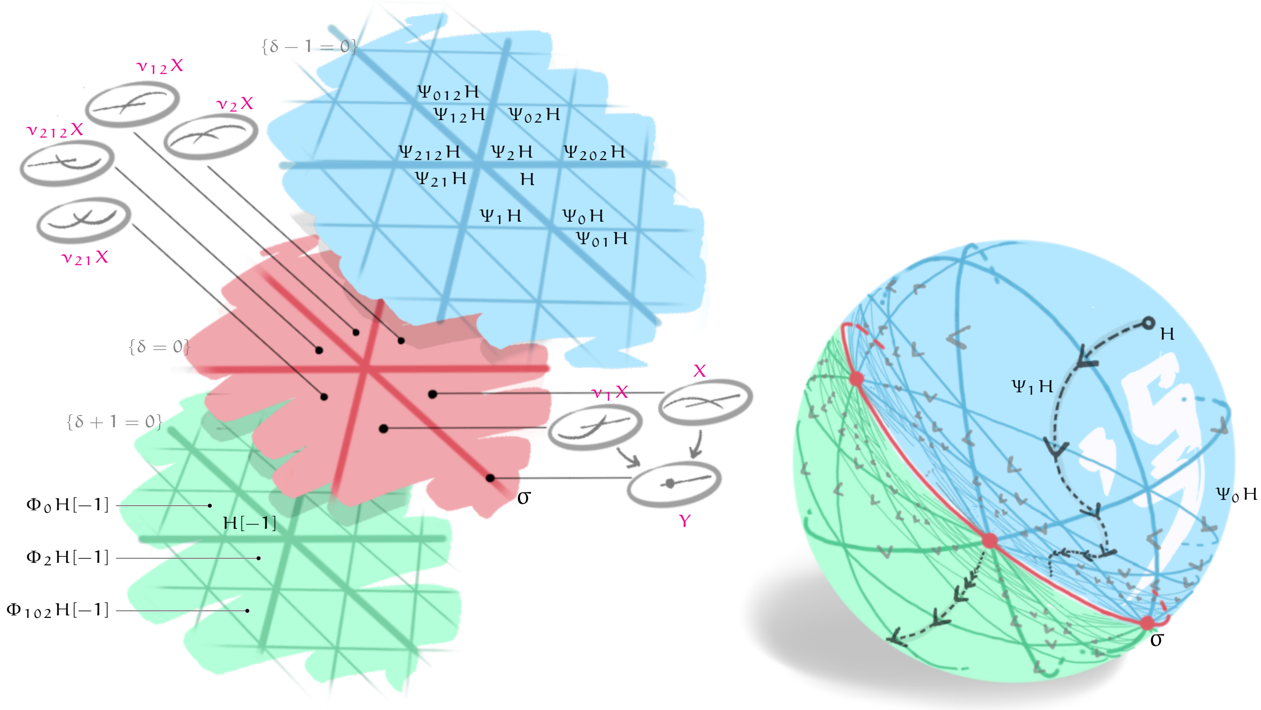


Figure 2. The 3-dimensional heart fan for a cA_2 resolution, sliced along affine hyperplanes (left) and the unit sphere (right). The mutation functors are abbreviated, e.g. by writing Ψ_{01} for $\Psi_1 \circ \Psi_0$. Cones in the hyperplane $\{\delta = 0\}$ have been labelled by the 3-fold whose geometric hearts they are associated to, where e.g. $\nu_{212}X$ is the flop of the second curve in $\nu_{12}X$.

Given a non-algebraic heart $\mathsf{K} \in \operatorname{tilt}(\mathcal{per}(\frac{X}{Z}))$, we can follow algorithm 1.1 to produce bounding sequences

$$\ldots\Psi_{i_2}\Psi_{i_1}\mathsf{H} \quad > \quad \mathsf{K} \quad > \quad \ldots\Phi_{j_2}\Phi_{j_1}\mathsf{H}[-1]$$

by iterated simple tilting. However this does not suffice for ‘pushing K to the geometric hyperplane’, and it is possible that the corresponding paths in the heart fan do not converge to the same cone (illustrated in fig. 2). In this regard the single-curve case is deceptively simple.

Establishing $\mathbf{C}\,\mathsf{K} \neq 0$ thus demands more finesse, and our strategy is to utilise line bundles on birational models for this purpose. Under the identification of $\operatorname{Pic}_{\mathbb{R}} X$ with the hyperplane $\{\delta = 0\}$, (the proper transform of) any line bundle $\mathcal{L} \in \operatorname{Pic} W$ corresponds to a vector θ in the heart fan, so for any cone σ we can consider the submonoid $\operatorname{Pic} W \cap \sigma$ (e.g. if $\sigma = \mathbf{C}(\operatorname{flop} \mathcal{coh}\, W)$ then this is the monoid of nef bundles). On the other hand, $\operatorname{Pic} W$ has a natural action on $\mathbf{D}^0 W$ which induces an action $\operatorname{Pic} W \circlearrowright \mathbf{D}^0 X$ across the composite flop equivalence. Using the abbreviated notation $\mathcal{L} \otimes \mathsf{H} \coloneqq \operatorname{flop}(\mathcal{L} \otimes_W (\operatorname{flop}^{-1} \mathsf{H}))$, we establish the following in § 6.

Theorem D (= 6.3, 6.4, 6.5). *Writing* $\mathsf{H} = \mathcal{per}(\frac{X}{Z})$ *for the standard heart in* $\mathbf{D}^0 X$*, the following statements hold.*

(1) Given any birational model W *of* X *and a line bundle* $\mathcal{L} \in \operatorname{Pic} W$*, any heart in* $\mathbf{D}^0 X$ *of the form* $\mathcal{L}^\vee \otimes \mathsf{H}$ *or* $\mathcal{L} \otimes \mathsf{H}[-1]$ *lies in* $\operatorname{tilt}(\mathsf{H})$ *if and only if* $\mathcal{L}$ *is nef.*

(2) If W' *is another birational model such that* $\mathcal{L} \in \operatorname{Pic} W$ *and its proper transform* $\mathcal{L}' \in \operatorname{Pic} W'$ *are both nef, then there are equalities of t-structures* $\mathcal{L}^\vee \otimes \mathsf{H} = \mathcal{L}'^\vee \otimes \mathsf{H}$, $\mathcal{L} \otimes \mathsf{H}[-1] = \mathcal{L}' \otimes \mathsf{H}[-1]$.

(3) For any cone $\sigma \subset \mathbf{C}(\operatorname{flop} \mathcal{coh}\, W)$ *in* $\operatorname{HFan}(\mathsf{H})$*, the induced actions of monoid* $\operatorname{Pic} W \cap \sigma$ *on the subsets*

$$\sigma\text{-tilt}^+(\mathsf{H}) = \bigcup_{\mathcal{L} \in \operatorname{Pic} W \cap \sigma} [\mathcal{L}^\vee \otimes \mathsf{H}, \mathsf{H}], \qquad \sigma\text{-tilt}^-(\mathsf{H}) = \bigcup_{\mathcal{L} \in \operatorname{Pic} W \cap \sigma} [\mathsf{H}[-1], \mathcal{L} \otimes \mathsf{H}[-1] \qquad \subseteq \operatorname{tilt}(\mathsf{H})$$

respect the partial order inherited from $\operatorname{tilt}(\mathsf{H})$ *and the monoid-order on* $\operatorname{Pic} W \cap \sigma$.

(4) Each heart in the posets above is algebraic, and conversely every algebraic heart in $\operatorname{tilt}(\mathsf{H})$ *lies in some poset of the above form for suitable* W, σ.

(5) The infimum of $\sigma\text{-tilt}^+(\mathsf{H})$ *is the maximal element of* $\operatorname{tilt}(\mathsf{H})$ *with heart cone* σ*, likewise the supremum of* $\sigma\text{-tilt}^-(\mathsf{H})$ *is the minimal element of* $\operatorname{tilt}(\mathsf{H})$ *with heart cone* σ.

The above result can be seen as a 'fixed-point theorem' for $\operatorname{tilt}(\mathsf{H})$. Indeed if $\mathcal{L} \in \operatorname{Pic} X$ is an ample line bundle, then the orbit of H under iterative applications of $\mathcal{L}^\vee \otimes (-)$ limits to $\mathcal{coh}\, X$ which is fixed as a heart by $\operatorname{Pic} X$.

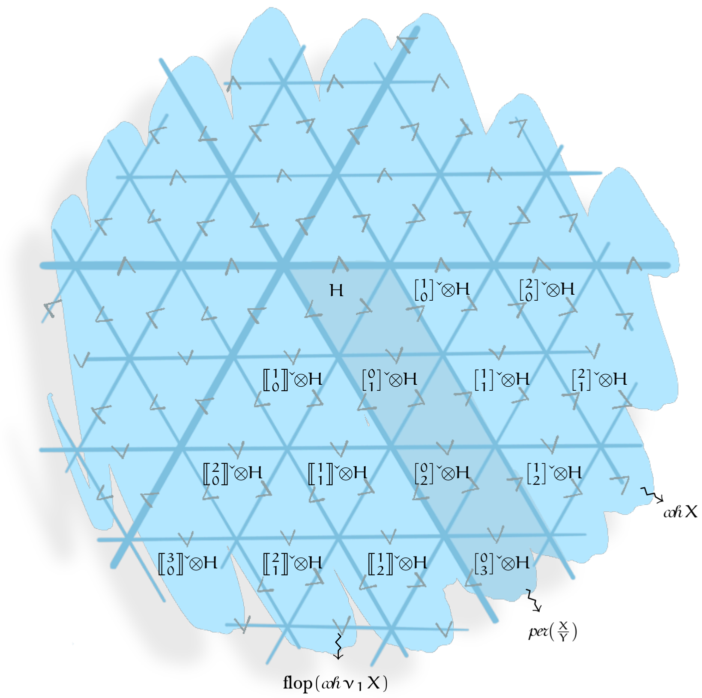


Figure 3. Continuing from fig. 2, the actions of $\operatorname{Pic} X$ and $\operatorname{Pic}(\nu_1 X)$ on H are shown. Here $[\begin{smallmatrix} i \\ j \end{smallmatrix}]$ denotes the line bundle on X which has degrees i and j on the two exceptional curves respectively, and we use double brackets ($[\![\begin{smallmatrix} i \\ j \end{smallmatrix}]\!]$) for bundles on the flop $\nu_1 X$.

Note that since $[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}]$ and its proper transform $[\![\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}]\!]$ are both trivial on the flopped curve, their actions coincide i.e. $[\![\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}]\!]^\vee \otimes \mathsf{H} = [\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}]^\vee \otimes \mathsf{H}$.

The shaded region represents $\sigma\text{-tilt}^+(\mathsf{H})$ for σ shown in fig. 2.

We then have the following recipe, which works generally to show every heart cone is non-zero.

Algorithm 1.2 (Skyscraper–hunting). Suppose $\mathsf{K} \subset \mathcal{per}(\frac{X}{Z})[-1,0]$ is a non-algebraic t-structure corresponding to a torsion pair $\mathcal{per}(\frac{X}{Z}) = \mathsf{T} * \mathsf{F}$.

1. By lemma 6.11 there is a unique birational model $W \in \mathrm{Bir}(\frac{X}{Z})$ such that $\mathrm{flop}^{-1}(\mathsf{K})$ is intermediate with respect to $\mathcal{per}(\frac{W}{Z})$ and also contains the full subcategory $\{w \in \mathcal{coh}\, W \mid \mathbf{R}\pi_* w = 0\}$ (we say K *lives* on W). Thus replacing K by $\mathrm{flop}^{-1}(\mathsf{K})$ if necessary, we may assume K lives on X.

   With this hypothesis satisfied, every skyscraper sheaf $\mathcal{O}_p \in \mathcal{per}(\frac{X}{Z}) \cap \mathcal{coh}\, X$ is either torsion or torsion-free with respect to the torsion pair $\mathsf{T} * \mathsf{F}$ (lemma 6.12).

2. Suppose $C_i \subset X$ is an exceptional curve and $\mathcal{L}_i \in \mathrm{Pic}\, X$ is the line bundle with degree 1 on C_i and degree 0 elsewhere. In particular, the maximal and minimal elements of $\mathrm{tilt}(\mathsf{H})$ whose heart cone is generated by $\mathcal{L}_i$ are the categories $\mathcal{per}\left(\frac{X}{X_i}\right)$ and $\overline{\mathcal{per}}\left(\frac{X}{X_i}\right)$ respectively, where $X \to X_i$ is the partial contraction of C_i.

   Now if some $p \in C_i$ satisfies $\mathcal{O}_p \in \mathsf{T}$, then lemma 6.7 shows $\mathcal{L}_i^{\otimes(-n)} \otimes \mathsf{H} > \mathsf{K}$ for arbitrarily large n, and thus $\mathcal{per}\left(\frac{X}{X_i}\right) \geqslant \mathsf{K}$. Likewise if some $p \in C_i$ satisfies $\mathcal{O}_p \in \mathsf{F}$, then we have the bound $\mathsf{K} \geqslant \overline{\mathcal{per}}\left(\frac{X}{X_i}\right)$.

3. Thus if there is some curve C_i with two points $p, q \in C_i$ such that $\mathcal{O}_p \in \mathsf{T}$ and $\mathcal{O}_q \in \mathsf{F}$, then K lies in the interval $\left[\mathcal{per}\left(\frac{X}{X_i}\right), \overline{\mathcal{per}}\left(\frac{X}{X_i}\right)\right]$ and in particular has non-zero heart cone.

4. Otherwise by connectivity of the exceptional curves in X, either T or F contains *all* the skyscrapers. By a similar argument, this forces one of the bounds $\mathsf{K} \geqslant \mathcal{coh}\, X$ or $\mathsf{K} \leqslant \overline{\mathcal{coh}}\, X$. The bound on the other side can then be found by chasing simple tilts from H or $\mathsf{H}[-1]$ as in algorithm 1.1, and this suffices to prove that $\mathbf{C}\,\mathsf{K}$ is non-zero (lemma 6.9).

Heart cones of semi-geometric hearts. Continuing to work with the cA_2 crepant resolution above, consider a flop $\rho : X \dashrightarrow \nu_1 X$, and the partial contraction $\tau : X \to Y$ of the flopped curve. The heart cone of the associated semi-geometric heart $\mathcal{per}(\frac{X}{Y})$ is given by the ray $\sigma = \mathbf{C}(\mathcal{coh}\, X) \cap \mathbf{C}(\mathrm{flop}\, \mathcal{coh}(\nu_1 X))$.

To examine other hearts that lie on σ, note that Y has a unique singular point (the image of the flopped curve), and ρ is simply the Atiyah flop over a neighbourhood Z_1 of this point. Computing the category of σ-semistable objects then recovers the category of perverse sheaves associated to the cA_1 flopping contraction $\tau^{-1} Z_1 \to Z_1$, and thus (up to tilting in skyscrapers in the smooth locus) every other intermediate heart with heart cone σ is obtained from tilts of this smaller category $\mathcal{per}\left(\frac{\tau^{-1} Z_1}{Z_1}\right)$.

Convex geometrically this manifests as the fact that the cA_2 intersection arrangement 'looks like' the cA_1 arrangement locally around σ (as is apparent from figs. 1 and 2). This phenomenon is more general and a correspondence between the heart fan of H and that of its semistable subcategories is sketched in remark 2.15. Broomhead–Pauksztello–Ploog–Woolf's *tangent multifan* construction [see BPPW24, 'Further work'] investigates this to a greater depth.

Notation and Conventions. We work over the ground field $\mathbb{C}$, and fix once and for all a complete local isolated compound du Val singularity $Z = \mathrm{Spec}\, R$ with singular point corresponding to the maximal ideal $\mathfrak{m} \subset R$.

All algebras we consider are finitely generated R-algebras. Given such an algebra Λ, we write $\mathbf{D}^b \Lambda$ for the bounded derived category of right Λ-modules and $\mathbf{D}^{fl} \Lambda$ for the full subcategory of complexes whose cohomology (with respect to $\mathcal{mod}\,\Lambda$) has finite length. This forms a triangulated subcategory of $\mathbf{D}^b \Lambda$, and we write $\mathbf{K}\,\Lambda$ for the $\mathbf{K}$-theory (i.e. Grothendieck group) of $\mathbf{D}^{fl} \Lambda$.

Likewise, all 3-folds we consider are Z-schemes. Given such a scheme $\pi : X \to Z$, we write $\mathbf{D}^b X$ for the bounded derived category of coherent sheaves on X and $\mathbf{D}^0 X$ for the full subcategory of $\mathbf{D}^b X$ containing objects supported on $\pi^{-1}[\mathfrak{m}]$. Again, we write $\mathbf{K}\, X$ for the Grothendieck group of $\mathbf{D}^0 X$.

When the map $\pi : X \to Z$ is a flopping contraction, the associated category $\mathcal{Per}(\frac{X}{Z})$ of perverse sheaves is central to our exposition. The definition depends on a choice of "perversity" $p \in \mathbb{Z}$, and we always set $p = 0$. Thus in Van den Bergh's [Van04] or Bridgeland's [Bri02] notation, the category $\mathcal{Per}(\frac{X}{Z})$ would be denoted ${}^0\mathrm{Per}(X/Z)$.

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§ 2 Torsion classes in an algebraic category

Let H be an Abelian category. We say H is *algebraic* if it is Artinian and Noetherian, and has finitely many simple objects. In particular the Jordan–Hölder theorem holds and each object of H admits a finite filtration by simple objects, and the associated graded object (a direct sum of simples) is independent of the choice of filtration. Consequently the Grothendieck group $\mathbf{K}\,\mathsf{H}$ is a free Abelian group of finite rank with basis given by the classes of simples.

Given subcategories $\mathsf{U}, \mathsf{V} \subset \mathsf{H}$ we write $\mathsf{U} * \mathsf{V}$ for the full subcategory of objects $h \in \mathsf{H}$ which sit in some exact sequence $0 \to u \to h \to v \to 0$ with $u \in \mathsf{U}$, $v \in \mathsf{V}$. The operation $(*)$ is associative, so writing $\mathsf{U} * \mathsf{V} * \mathsf{W}$ for $\mathsf{U}, \mathsf{V}, \mathsf{W} \subset \mathsf{H}$ is unambiguous. Then we can define the *extension closure* of $\mathsf{U} \subset \mathsf{H}$ as

$$\langle \mathsf{U} \rangle = \bigcup_{n \geqslant 0} \underbrace{\mathsf{U} * \ldots * \mathsf{U}}_{n \text{ factors}}, \tag{1}$$

and say U is closed under extensions (or *extension-closed*) if $\langle \mathsf{U} \rangle = \mathsf{U}$. Equivalently, $\langle \mathsf{U} \rangle$ is the full subcategory of objects which admit a filtration by objects of U.

For a subcategory $\mathsf{U} \subset \mathsf{H}$, we define its left orthogonal complement ${}^{\perp}\mathsf{U}$ as the full subcategory of objects $h \in \mathsf{H}$ such that $\mathrm{Hom}(h, u) = 0$ for all $u \in \mathsf{U}$. The right orthogonal complement $\mathsf{U}^{\perp}$ is defined analogously. It can be shown that both $\mathsf{U}^{\perp}$ and ${}^{\perp}\mathsf{U}$ are always extension-closed.

Definition 2.1. We say a subcategory $\mathsf{T} \subset \mathsf{H}$ is a *torsion class* if it satisfies $\mathsf{T} = {}^{\perp}(\mathsf{T}^{\perp})$, and write $\mathrm{tors}(\mathsf{H})$ for the collection of all torsion classes in H. Dually, we say $\mathsf{F} \subset \mathsf{H}$ is a *torsion-free class* if $\mathsf{F} = ({}^{\perp}\mathsf{F})^{\perp}$ holds and we write $\mathrm{torf}(\mathsf{H})$ for the collection of all torsion-free classes in H.

For any $\mathsf{U} \subset \mathsf{H}$ it can be shown that ${}^{\perp}\mathsf{U}$ is a torsion class while $\mathsf{U}^{\perp}$ is a torsion-free class. In fact the assignment $\mathsf{F} \mapsto {}^{\perp}\mathsf{F}$ is a bijection $\mathrm{torf}\,\mathsf{H} \to \mathrm{tors}\,\mathsf{H}$ with the inverse map given by $\mathsf{T} \mapsto \mathsf{T}^{\perp}$. Further, this correspondence is such that $\mathsf{H} = \mathsf{T} * \mathsf{T}^{\perp}$ whenever T is a torsion class. Hence we write '$\mathsf{H} = \mathsf{T} * \mathsf{F}$ *is a torsion pair*' to mean $\mathsf{T} \subset \mathsf{H}$ is a torsion class with corresponding torsion-free class $\mathsf{F} = \mathsf{T}^{\perp}$.

Torsion and torsion-free classes are always closed under extensions. Further, torsion classes are closed under factors (i.e. if T is a torsion class and we have $h \in \mathsf{T}$ then we also have $h' \in \mathsf{T}$ whenever there is a surjection $h \twoheadrightarrow h'$). Dually, torsion-free classes are closed under sub-objects. The converse implications hold when H satisfies additional hypotheses– if H is Noetherian then a subcategory $\mathsf{T} \subset \mathsf{H}$ that is closed extensions and factors is a torsion class, dually if H is Artinian then any subcategory closed under extensions and sub-objects is a torsion-free class.

§ 2.1 Recollections on t-structures. Let $\mathcal{T}$ be a triangulated category with shift functor $[1]$. For subcategories $\mathsf{U}, \mathsf{V} \subset \mathcal{T}$, we define the subcategory $\mathsf{U} * \mathsf{V}$ analogously with exact triangles instead of short exact sequences. The operation $(*)$ is associative on subcategories [BBD82, lemma 1.3.10], so the extension closure is defined just as in (1). Likewise, the orthogonal complements ${}^{\perp}\mathsf{U}$ and $\mathsf{U}^{\perp}$ are defined as full subcategories of objects with no morphisms into (resp. from) any object of U.

Given an interval $I \subset \mathbb{Z}$ and a subcategory $\mathcal{U} \subset \mathcal{T}$, we write $\mathcal{U}[I]$ for the extension-closure of $\{u[i] \mid u \in \mathcal{U},\ i \in I\}$. We use obvious notational choices whenever convenient, so $\mathcal{U}[\leqslant 0] = \mathcal{U}[(-\infty, 0]]$ and $\mathcal{U}[0,1] = \mathcal{U}[[0,1]]$. Note for an interval $I = [i,j]$, we have $\mathcal{U}[i,j] = \mathcal{U}[j] * \mathcal{U}[j-1] * \ldots * \mathcal{U}[i]$.

Definition 2.2. A full additive subcategory $\mathcal{H} \subset \mathcal{T}$ is the *heart of a (bounded) t-structure* if it is closed under extensions, $\mathcal{H}[\mathbb{Z}] = \mathcal{T}$, and $\mathcal{H}[\leqslant 0] = (\mathcal{H}[>0])^{\perp}$. We write t-str$(\mathcal{T})$ for the collection of all hearts of bounded t-structures in $\mathcal{T}$.

We remark that each of the three subcategories $\mathcal{H}[\leqslant 0]$, $\mathcal{H}[>0]$, $\mathcal{H}$ (called the *aisle*, *coaisle*, and the *heart* respectively) determines the other two and hence saying "the t-structure $\mathcal{H}[\leqslant 0]$" is unambiguous.

The heart $\mathcal{H}$ of a t-structure is always Abelian, with $0 \to h' \to h \to h'' \to 0$ exact if and only if the corresponding triangle $h' \to h \to h'' \to h'[1]$ is distinguished in $\mathcal{T}$. Further, the natural map of Grothendieck groups $\mathbf{K}\,\mathcal{U} \to \mathbf{K}\,\mathcal{T}$ is an isomorphism. We say a t-structure is *algebraic* if its heart is algebraic as an Abelian category.

Further, if $\mathcal{H} \subset \mathcal{T}$ is the heart of a t-structure then there are additive functors $\mathrm{H}^i : \mathcal{T} \to \mathcal{H}$ ($i \in \mathbb{Z}$), called the *cohomology functors with respect to* $\mathcal{H}$, such that any $t \in \mathcal{T}$ is filtered by the objects $\left\{\mathrm{H}^i(t)[-i] \mid i \in \mathbb{Z}\right\}$ finitely many of which are non-zero. In particular, for an interval $I \subset \mathbb{Z}$ we see that $\mathcal{H}[I]$ contains precisely those objects t for which $\mathrm{H}^{-i}(t) = 0$ whenever $i \notin I$. We say such objects t are *concentrated in cohomological degrees in* I.

Objects concentrated in cohomological degrees -1 and 0 with respect to $\mathcal{H}$, the so-called *two-term complexes*, are of special interest since their properties depend only on $\mathcal{H}$ as an Abelian category. We say a t-structure containing only such objects is *intermediate with respect to* $\mathcal{H}$, and write tilt$(\mathcal{H}) \subset$ t-str$(\mathcal{T})$ for the collection of intermediate t-structures, namely

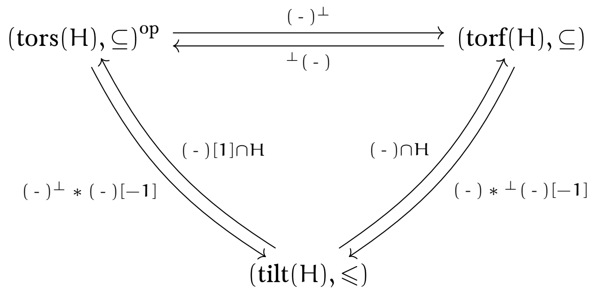
$$\mathrm{tilt}(\mathcal{H}) = \{\mathcal{K} \in \text{t-str}(\mathcal{T}) \mid \mathcal{K} \subset \mathcal{H}[-1,0]\}.$$

The notation is explained by the fact that any $\mathcal{K} \in \mathrm{tilt}(\mathcal{H})$ can be obtained from $\mathcal{H}$ by the process of *tilting* as described in [HRS96], who show that in this case there is a torsion pair $\mathcal{H} = \mathcal{T} * \mathcal{F}$ on $\mathcal{H}$ such that $\mathcal{K} = \mathcal{F} * \mathcal{T}[-1]$. This torsion pair is obtained as $\mathcal{T} = \mathcal{K}[1] \cap \mathcal{H}$, $\mathcal{F} = \mathcal{K} \cap \mathcal{H}$, and the correspondence is bijective, and we say that $\mathcal{K}$ is the *(negative) tilt of* $\mathcal{H}$ *along the given torsion pair*. Given this correspondence, it follows that tilt$(\mathcal{H})$ depends only on $\mathcal{H}$ as an Abelian category and not on the ambient triangulated category.

We write alg-tilt$(\mathcal{H})$ for the set of algebraic t-structures that are intermediate with respect to $\mathcal{H}$, and ftors$(\mathcal{H})$, ftorf$(\mathcal{H})$ for the corresponding torsion (resp. torsion-free) classes. When $\mathcal{H}$ is itself algebraic, the torsion theories corresponding to algebraic tilts are called *functorially finite* and can be characterised in multiple ways (see for example [Asa20].)

§ 2.2 The lattice theory of torsion classes. Again, let $\mathcal{T}$ be a triangulated category with a t-structure $\mathcal{H} \subset \mathcal{T}$. The set of t-structures t-str$(\mathcal{T})$ can be assigned a partial order given by inclusion of aisles, i.e. we write $\mathcal{K}' \leqslant \mathcal{K}$ whenever we have $\mathcal{K}'[\leqslant 0] \subseteq \mathcal{K}[\leqslant 0]$. The subset tilt$(\mathcal{H})$ is then precisely the interval $[\mathcal{H}[-1], \mathcal{H}]$ and in particular inherits this partial order. If we equip tors$(\mathcal{H})$, torf$(\mathcal{H})$ with their natural (inclusion) orders then the correspondences between these sets can be upgraded to a commutative diagram of poset isomorphisms

(2)



and these restrict to isomorphisms between the sub-posets ftors$(\mathcal{H})^{\mathrm{op}}$, ftorf$(\mathcal{H})$, and alg-tilt$(\mathcal{H})$.

When $\mathcal{H}$ is algebraic, each poset appearing in (2) is a *complete lattice*, i.e. it admits arbitrary infima (greatest lower bounds) and suprema (least upper bounds), since the intersection of torsion classes remains a torsion class and this gives the infima, while the existence of suprema follows from the poset isomorphism with $(\operatorname{torf}\mathcal{H})^{op}$ and observing that the intersection of torsion-free classes remains torsion-free. Explicitly, given a collection of torsion classes $\{\mathcal{T}_i \mid i \in I\} \subset \operatorname{tors}(\mathcal{H})$ we have

$$\inf_{i\in I} \mathcal{T}_i = \bigcap_{i\in I} \mathcal{T}_i, \qquad \sup_{i\in I} \mathcal{T}_i = \left\langle \left\{ h \in \mathcal{H} \mid \text{there is a surjection } t \twoheadrightarrow h \text{ for some } t \in \bigcup_{i\in I} \mathcal{T}_i \right\} \right\rangle.$$

We assume $\mathcal{H}$ is algebraic for the rest of this subsection.

Recall that we say a *covers* b (written $a \gtrdot b$) in a partially ordered set if $a > b$ and there is no element c satisfying $a > c > b$. The *Hasse quiver* of a poset has vertices given by the elements, and an arrow $a \to b$ for each covering relation $a \gtrdot b$. The join and meet operations on lattices of torsion classes are semidistributive so their Hasse quivers are naturally labelled by certain indecomposable objects [see BCZ19], we now describe the construction.

Definition 2.3. An object $b \in \mathcal{H}$ is a *brick* if all non-zero endomorphisms of b are invertible. A *semibrick* $\mathcal{S} \subset \mathcal{H}$ is a set of bricks that are pairwise orthogonal, i.e. each $b \in \mathcal{S}$ is a brick, and whenever $b_1, b_2 \in \mathcal{S}$ are non-isomorphic bricks, we have $\operatorname{Hom}(b_1, b_2) = 0$.

By Schur's lemma, every simple object in $\mathcal{H}$ is a brick and the collection of all simples forms a semibrick. It is not hard to see that if $s \in \mathcal{H}$ is simple, then $\mathcal{T} = \langle s \rangle$ is a minimal non-zero torsion class in $\mathcal{H}$ (i.e. the relation $0 \subset\!\!\cdot\ \mathcal{T}$ is covering in $\operatorname{tors}(\mathcal{H})$), and any torsion-free class covering 0 must be of this form. Likewise every maximal proper torsion class (i.e. a torsion class $\mathcal{U}$ such that the relation $\mathcal{U} \subset\!\!\cdot\ \mathcal{H}$ is covering) is precisely of the form ${}^{\perp}s$ for some simple object $s \in \mathcal{H}$. Intermediate hearts arising from such torsion theories are called *simple tilts.*

In fact all covering relations arise in this way from simple tilts.

Theorem 2.4 [DIRRT23, theorems 3.3–3.4]. *If* $\mathcal{K}' \lessdot \mathcal{K}$ *is a covering relation in* $\operatorname{tilt}(\mathcal{H})$*, then* $(\mathcal{K} \cap \mathcal{H}) \setminus \mathcal{K}'$ *contains a unique brick* b *which we call the* brick-label *of the covering relation. This brick is a simple object of* $\mathcal{K}$*, and the corresponding simple tilt in the torsion class* $\langle b \rangle \in \operatorname{tors}(\mathcal{K})$ *is* $\mathcal{K}'$.

More generally if $\mathcal{K}' < \mathcal{K}$ *are intermediate t-structures with respect to* $\mathcal{H}$ *then the set of bricks in* $(\mathcal{K} \cap \mathcal{H}) \setminus \mathcal{K}'$ *is non-empty and equals the set of brick-labels of covering relations in the interval* $[\mathcal{K}', \mathcal{K}] \subset \operatorname{tilt}(\mathcal{H})$.

Thus each arrow in the Hasse quiver of $\operatorname{tilt}(\mathcal{H})$ is labelled by a unique brick in $\mathcal{H}$, and each brick in $\mathcal{H}$ arises as the brick-label of at least one such arrow.

The posets $\operatorname{tors}(\mathcal{H})$ and $\operatorname{torf}(\mathcal{H})$ inherit this labelling across the isomorphisms (2). In particular if the covering relation $\mathcal{T}' \supset\!\!\cdot\ \mathcal{T}$ in $\operatorname{tors}(\mathcal{H})$ has brick-label b, then b is the unique brick in $\mathcal{T}' \setminus \mathcal{T}$ and we have $\mathcal{T}' \cap \mathcal{T}^{\perp} = \langle b \rangle$, $\mathcal{T} = {}^{\perp}b \cap \mathcal{T}'$, and $\mathcal{T}' = \mathcal{T} * \langle b \rangle$. The analogous statement holds for covering relations of torsion-free classes.

§ 2.3 Widely generated torsion theories. Given an algebraic Abelian category $\mathcal{H}$, we review how the lattice of torsion classes can be employed to study the set $\operatorname{sbrick}(\mathcal{H})$ of semibricks in $\mathcal{H}$.

Note that if $\mathcal{K} \subset \mathcal{H}[-1, 0]$ is an intermediate heart, then every simple object of $\mathcal{K}$ lies in a single cohomological degree with respect to $\mathcal{H}$. Thus writing $\operatorname{simp}(\mathcal{K})$ for the set of simple objects of $\mathcal{K}$, we see that $\operatorname{simp}(\mathcal{K})[1] \cap \mathcal{H}$ is a semibrick in $\mathcal{H}$, and this semibrick is contained in the torsion class associated to $\mathcal{K}$. This defines a map $\operatorname{simp}(-)[1] \cap \mathcal{H} : \operatorname{tilt}(\mathcal{H}) \to \operatorname{sbrick}(\mathcal{H})$, equivalently a map $\operatorname{tors}(\mathcal{H}) \to \operatorname{sbrick}(\mathcal{H})$ which maps any torsion class to a semibrick inside it.

On the other hand each $\mathcal{S} \in \operatorname{sbrick}(\mathcal{H})$ defines a torsion-free class $\mathcal{F} = \mathcal{S}^{\perp}$, with corresponding torsion class $\mathcal{T} = {}^{\perp}(\mathcal{S}^{\perp})$ characterised by the property of being minimal among all torsion classes containing $\mathcal{S}$. We say the torsion class $\mathcal{T}$ in this case is *generated by* $\mathcal{S}$. This defines a map $\operatorname{sbrick}(\mathcal{H}) \to \operatorname{tors}(\mathcal{H})$, equivalently a map $\operatorname{sbrick}(\mathcal{H}) \to \operatorname{tilt}(\mathcal{H})$ which we show is a section of $\operatorname{simp}(-)[1] \cap \mathcal{H}$.

Lemma 2.5. *Suppose* $H = T * F$ *is a torsion pair such that the torsion class* T *is generated by a semibrick* $S \subseteq H$. *Writing* $K = F * T[-1]$ *for the corresponding tilt, then an object* $k \in K \cap H[-1]$ *is simple in* K *if and only if it lies in* $S[-1]$.

Proof. Given $k \in S[-1]$, suppose there is an injection $k' \hookrightarrow k$ in K, with $k' \neq 0$. Since $T[-1] \subset K$ is a torsion-free class, k' also lies in $T[-1]$. Thus the object $k'[1] \in T$ is filtered by objects in S and their factors in H [see for example MS17, lemma 3.1]. In particular, there is some $s \in S$ with a non-zero morphism $s \to k'[1]$.

By injectivity of $k' \hookrightarrow k$, the composite map $s[-1] \to k' \to k$ is non-zero. Since S is a semibrick, it follows that $s[-1] \cong k$ and this is a splitting of the injection $k' \hookrightarrow k$. But k is indecomposable (since $k[1]$ is a brick), so the map $k' \to k$ is an isomorphism. Thus k has no non-trivial sub-objects in K, i.e. k is simple as required.

Conversely suppose $k \in K \cap H[-1]$ is simple in K. It follows that $k[1]$ is the quotient (in H) of some $s \in S$. In other words, there is an $h \in H$ and an exact triangle $h \to s \to k[1] \to h[1]$. Claim $h = 0$, so that $k[1] = s$ lies in S as required.

To prove the claim, first note that the triangle $s[-1] \to k \to h \to s$ shows that h cannot be a non-zero object in K. Thus if h is non-zero, then considering the torsion part of h shows that there is a non-zero composite map $s' \to h \to s$ for some $s' \in S$. But S is a semibrick, so a similar argument as above shows that the map $h \to s$ is an isomorphism and $k = 0$, which is a contradiction. □

It follows that assigning a semibrick $S \subseteq H$ to the the minimal torsion class it generates gives an injective map $\mathrm{sbrick}(H) \to \mathrm{tors}(H)$. The following proposition provides lattice-theoretic and homological characterisations of the image of this map, and provides alternative ways to read off a semibrick S from the torsion class it generates.

Proposition 2.6. *Given a torsion pair* $H = T * F$ *with corresponding tilt* $K = F * T[-1]$, *the following are equivalent.*

(1) The torsion class T *is generated by a semibrick* $S \subseteq H$.

(2) The interval $[0, T] \subset \mathrm{tors}(H)$ *is* coatomic, *i.e. for every* $U \in [0, T)$ *there is a* $U' \in [0, T)$ *with* $U \subseteq U' \lessdot T$.

(3) In K, *every non-zero object has a simple sub-object.*

If the above statements hold, then the semibrick S *which generates* T *is unique and is determined as*

$$\begin{aligned} S &= \{b \in H \mid b \textit{ is the brick-label of a relation } U \lessdot T \textit{ in } \mathrm{tors}(H)\} \\ &= \{b \in H \mid b[-1] \textit{ is a simple object of } K\}. \end{aligned}$$

We have seen (lemma 2.5) that if (1) holds, then the semibrick S which generates T is unique and is given by $\mathrm{simp}(K)[1] \cap H$. The equivalence (1) $\iff$ (2) is the content of [AP22, theorem 7.2], which also shows that in this case T is generated by the set $\{b \in H \mid b \text{ is the brick-label of a relation } U \lessdot T \text{ in } \mathrm{tors}(H)\}$. Since this set is a semibrick (see theorem 4.2 *ibid.*), it must coincide with S.

It thus remains to show the equivalence (1) $\iff$ (3), which we now do.

Proof of proposition 2.6 (1)⇒(3). Suppose T is generated by a semibrick S, and $k \in K$ is a non-zero object with no simple sub-object. In particular any sub-object $k' \hookrightarrow k$ shares this property (i.e. k' has no simple sub-object), and there is at least one such proper non-zero sub-object k'.

Now no object of $S \subset \mathrm{simp}(K)$ can map to k, so k lies in $T[-1]^{\perp}$. Thus neither k nor k' lie in $T[-1]$, so passing to sub-objects if necessary, we may in fact assume k' and k lie in F. But $F \subset K$ is a torsion class, so the cokernel k/k' of this morphism also lies in F. It follows that the map $k' \to k$ is also a proper injection in H.

But repeating the argument, this produces a chain of proper injections $\dots \hookrightarrow k'' \hookrightarrow k' \hookrightarrow k$ in H, which is a contradiction since H is Artinian. Thus every non-zero $k \in K$ necessarily has a simple sub-object. □

Proof of proposition 2.6 (3)⇒(1). Suppose (3) holds. We show that any torsion-free class that is larger than F must contain some element of the semibrick $\mathsf{S} = \mathrm{simp}(\mathsf{K})[1] \cap \mathsf{H}$. It follows that T is the minimal torsion class containing S, i.e. T is generated by S as required.

Thus consider any torsion pair $\mathsf{H} = \mathsf{T} * \mathsf{F}$ such that $\mathsf{F} \subsetneq \mathsf{F}'$. Thus $\mathsf{T} \cap \mathsf{F}'$ is a non-zero torsion-free class in the Abelian category $\mathsf{K}[1] = \mathsf{F}[1] * (\mathsf{T} \cap \mathsf{T}') * (\mathsf{T} \cap \mathsf{F}')$, in particular $\mathsf{T} \cap \mathsf{F}'$ is closed under taking sub-objects in this category. But by hypothesis on K (equivalently $\mathsf{K}[1]$), any non-zero object in $\mathsf{T} \cap \mathsf{F}'$ has a simple sub-object which therefore also lies in $\mathsf{T} \cap \mathsf{F}'$. Thus F' has non-trivial intersection with the set $\mathrm{simp}(\mathsf{K}[1]) \cap \mathsf{H} = \mathsf{S}$, as required. □

The obvious dual statements to lemma 2.5 and proposition 2.6 hold. In particular if $\mathsf{H} = \mathsf{T} * \mathsf{F}$ is a torsion pair, then F is generated by a semibrick S if and only if every non-zero object in the tilt $\mathsf{K} = \mathsf{F} * \mathsf{T}[-1]$ has a simple factor, and in this case the semibrick generating F is determined as $\mathsf{S} = \mathrm{simp}(\mathsf{K}) \cap \mathsf{H}$.

Remark 2.7. If $W \subseteq \mathsf{H}$ is a ***wide*** subcategory (i.e. W is closed under extensions, kernels, and cokernels, and is therefore Abelian), then the set of simples of W is evidently a semibrick of H. Ringel [Rin76, §1.2] shows that every wide subcategory of H is in fact the extension-closure of its simples, and conversely the extension closure of any semibrick is a wide subcategory. Thus torsion(-free) classes in H that are generated by a semibrick are called ***widely generated*** [see Asa20; AP22; BCZ19; MS17].

By proposition 2.6 and its dual, any torsion (torsion-free) class in H associated to an Artinian (resp. Noetherian) tilted heart is widely generated. However, chain conditions on an Abelian category in general strictly stronger than the requirement that every non-zero object admit a simple sub-object or factor. Indeed in the setting of a 3-fold flopping contraction $\pi : X \to Z$, we show that the algebraic heart $\mathsf{H} = \mathit{per}(\frac{X}{Z})$ has tilts which exhibit each of the following behaviours (see § 5.3 and remark 5.12 for relevant constructions).

Tilted heart	Chain conditions... Artinian?	Chain conditions... Noetherian?	Widely generated... torsion class?	Widely generated... torsion-free class?
Any algebraic tilt, e.g. H	✓	✓	✓	✓
Any geometric tilt, e.g. $\mathit{coh}\, X$	✗	✓	✗	✓
Any reversed-geometric tilt, e.g. $\overline{\mathit{coh}}\, X$	✓	✗	✓	✗
Tilt of $\mathit{coh}\, X$ in $\langle \mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_n} \rangle$	✗	✗	✓	✓
Tilt of $\mathit{coh}\, X$ in $\langle \mathcal{O}_{p_1} \rangle$	✗	✗	✗	✓
Tilt of $\mathit{coh}\, X$ in $\langle \mathcal{O}_p \mid p \in C_1 \cup \{p_2, \ldots, p_n\} \rangle$	✗	✗	✓	✗
Tilt of $\mathit{coh}\, X$ in $\langle \mathcal{O}_p \mid p \in C_1 \rangle$	✗	✗	✗	✗

If we further assume that the exceptional fiber $\pi^{-1}[\mathfrak{m}]$ has $n \geqslant 3$ integral components $C_1, C_2, \ldots, C_n$ (indexed such that $C_1 \cap C_n = \emptyset$), then choosing closed points $p_i \in C_i$ $(i = 1, \ldots, n)$ allows us to construct hearts in $\mathsf{H}[-1, 0]$ which exhibit the following additional behaviours (the last four rows of the table above).

Every other tilt of H fits into one of the seven categories above. In fact ***any*** tilt of ***any*** algebraic Abelian category fits into one of the seven categories above; in particular if a torsion pair $\mathsf{H} = \mathsf{T} * \mathsf{F}$ is such that both T and F are widely generated then it is straightforward to show that the heart $\mathsf{K} = \mathsf{F} * \mathsf{T}[-1]$ is Artinian if and only if it is Noetherian.

§ 2.4 Heart fans and numerical torsion theories. Given a triangulated category $\mathfrak{T}$ with a t-structure $\mathsf{H} \subset \mathfrak{T}$, the poset of intermediate hearts with respect to H can be encoded into convex-geometric data, possibly incurring the loss of some information. To explain the construction, first fix a surjection $\mathbf{K}\mathfrak{T} \twoheadrightarrow \mathfrak{h}$ of the Grothendieck group onto a free Abelian group $\mathfrak{h}$ of finite rank. Taking $\mathbb{R}$-linear duals, note that the finite dimensional Euclidean space $\Theta = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}, \mathbb{R})$ thus injects into $\mathrm{Hom}_{\mathbb{Z}}(\mathbf{K}\mathfrak{T}, \mathbb{R})$ and hence any vector $\theta \in \Theta$ can be regarded as an $\mathbb{R}$-linear functional on $\mathbf{K}\mathfrak{T}$.

We say a subset of Θ is *almost rational* if it can be defined by inequalities with coefficients in $\mathfrak{h}$, and we say it is *rational* if it can be defined by finitely many such inequalities. For example, given any $\alpha \in \mathfrak{h}$ the hyperplane $\{\theta \in \Theta \mid \theta(\alpha) = 0\}$ and the closed half-space $\{\theta \in \Theta \mid \theta(\alpha) \geqslant 0\}$ determined by α are rational.

We use obvious notational shorthands when convenient, for example $\{\alpha, \beta \geqslant 0\}$ denotes the rational subset $\{\theta \in \Theta \mid \theta(\alpha) \geqslant 0 \text{ and } \theta(\beta) \geqslant 0\}$.

A *cone* in Θ for us is a closed, convex, almost rational subset $\sigma \subset \Theta$ such that whenever we have vectors $\theta, \theta' \in \sigma$ and non-negative reals $a, a' \geqslant 0$, we also have $a\theta + a'\theta' \in \sigma$. If a cone σ lies in the half-space $\{\alpha \geqslant 0\}$, we say the subset $\sigma \cap \{\alpha = 0\}$ is a *face* of σ. We write $\text{faces}(\sigma)$ for the collection of all faces of σ.

Definition 2.8. We say a collection of cones Σ is a *fan* in Θ if it is closed under taking faces (i.e. if $\sigma \in \Sigma$ then $\text{faces}(\sigma) \subset \Sigma$) and any two cones in Σ intersect only in faces (i.e. if $\sigma, \sigma' \in \Sigma$ then $\sigma \cap \sigma' \in \text{faces}(\sigma)$). We say the fan Σ is is *complete* if every vector $\theta \in \Theta$ is in some cone $\sigma \in \Sigma$, and Σ is *simplicial* if each cone $\sigma \in \Sigma$ is simplicial (i.e. generated by linearly independent vectors).

For any subcategory $\mathsf{U} \subset \mathfrak{T}$, the *dual cone of* U (with respect to $\mathfrak{h}$) is defined to be

$$\mathbf{C}(\mathsf{U}) = \{\theta \in \Theta \mid \theta[u] \geqslant 0 \text{ for all } u \in \mathsf{U}\}.$$

The following lemma is immediate from the constructions, and is useful when relating heart cones of intermediate hearts to torsion theories.

Lemma 2.9. *If* K *is the tilt of* H *in a torsion pair* $\mathsf{H} = \mathsf{T} * \mathsf{F}$, *then* $\mathbf{C}(\mathsf{K}) = \mathbf{C}\,\mathsf{F} \cap (-\mathbf{C}\,\mathsf{T})$ *as subsets of* Θ.

We also make the useful observation that dual cones transform via an 'inverse–transpose' rule under functorial equivalences.

Lemma 2.10. *Suppose* $\Phi : \mathfrak{T}_1 \to \mathfrak{T}_2$ *is an exact equivalence of triangulated categories and the finite-rank free Abelian quotients* $\mathbf{K}\,\mathfrak{T}_i \twoheadrightarrow \mathfrak{h}_i$ *(*$i = 1, 2$*) are such that* Φ *induces a linear isomorphism* $\varphi : \mathfrak{h}_1 \to \mathfrak{h}_2$. *Then the corresponding isomorphism* $(\varphi^{\vee})^{-1} : \text{Hom}(\mathfrak{h}_1, \mathbb{R}) \to \text{Hom}(\mathfrak{h}_2, \mathbb{R})$ *governs the transformation of dual cones, i.e. for any* $\mathsf{U} \subset \mathfrak{T}_1$ *we have* $\mathbf{C}(\Phi\mathsf{U}) = (\varphi^{\vee})^{-1}\,\mathbf{C}(\mathsf{U})$.

Now if $\mathsf{K} \subset \mathfrak{T}$ is the heart of a t-structure, the dual cone $\mathbf{C}(\mathsf{K})$ is also called the *heart cone*. Broomhead, Pauksztello, Ploog, and Woolf [BPPW24] show that the heart cones coming from $\text{tilt}(\mathsf{H})$ form a fan in Θ, which generalises various well-known constructions of fans coming silting theory and the theory of stability conditions.

Theorem 2.11 [BPPW24, theorem A]. *For* H, $\mathfrak{T}$, $\mathfrak{h}$, *and* Θ *as above, the collection of cones given by*

$$\text{HFan}(\mathsf{H}) = \bigcup_{\mathsf{K} \in \text{tilt}(\mathsf{H})} \text{faces}\,(\mathbf{C}\,\mathsf{K})$$

is a fan in Θ, *called the* heart fan *of* H *(with respect to* $\mathfrak{h}$*). If* H *is algebraic, then this fan is complete and simplicial. If in addition the map* $\mathbf{K}\,\mathfrak{T} \twoheadrightarrow \mathfrak{h}$ *is an isomorphism, then an intermediate heart* K *is algebraic if and only if* $\mathbf{C}(\mathsf{K})$ *is a full-dimensional cone, and in this case* K *is the unique intermediate heart with heart cone* $\mathbf{C}(\mathsf{K})$.

For the rest of this subsection, assume H is algebraic and that the map $\mathbf{K}\,\mathfrak{T} \twoheadrightarrow \mathfrak{h}$ is an isomorphism. Under these assumptions, we have plenty of control over the natural map

$$\mathbf{C} : \text{tilt}(\mathsf{H}) \to \text{HFan}(\mathsf{H}). \tag{3}$$

For instance, by the last part of theorem 2.11, the fiber of this map over a full dimensional cone contains a single element. However, in general, one can have a multiple torsion classes giving the same heart cone. It is possible to describe the complete fiber of $\mathbf{C}$ over any non-zero cone by considering the *numerical torsion theories* defined in [BKT14]. To define these, given $\theta \in \Theta$ we consider the subcategories

$$\begin{aligned}
\mathsf{H}^{\text{tr}}(\theta) &= \{h \in \mathsf{H} \mid \theta[f] \leqslant 0 \text{ for all factors } h \twoheadrightarrow f\},\\
\mathsf{H}_{\text{tr}}(\theta) &= \{h \in \mathsf{H} \mid \theta[f] < 0 \text{ for all non-zero factors } h \twoheadrightarrow f \neq 0\},\\
\mathsf{H}^{\text{tf}}(\theta) &= \{h \in \mathsf{H} \mid \theta[s] \geqslant 0 \text{ for all sub-objects } s \hookrightarrow h\},\\
\mathsf{H}_{\text{tf}}(\theta) &= \{h \in \mathsf{H} \mid \theta[s] > 0 \text{ for all non-zero sub-objects } 0 \neq s \hookrightarrow h\}.
\end{aligned}$$

These define two torsion pairs $\mathsf{H} = \mathsf{H}^{\mathrm{tr}}(\theta) * \mathsf{H}_{\mathrm{tf}}(\theta) = \mathsf{H}_{\mathrm{tr}}(\theta) * \mathsf{H}^{\mathrm{tf}}(\theta)$, and we call the corresponding tilts $\mathsf{H}_{\mathrm{tt}}(\theta), \mathsf{H}^{\mathrm{tt}}(\theta)$ respectively. Evidently $\mathsf{H}_{\mathrm{tt}}(\theta) \leqslant \mathsf{H}^{\mathrm{tt}}(\theta)$, and we have an interval $[\mathsf{H}_{\mathrm{tt}}\theta, \mathsf{H}^{\mathrm{tt}}\theta] \subset \mathrm{tilt}(\mathsf{H})$. This interval contains precisely the intermediate hearts K which satisfy $\theta \in \mathbf{C}(\mathsf{K})$.

Lemma 2.12. *Let* $\mathsf{K} \in \mathrm{tilt}(\mathsf{H})$ *be an intermediate heart, and fix a vector* $\theta \in \Theta$. *Then we have* $\theta \in \mathbf{C}(\mathsf{K})$ *if and only* $\mathsf{H}_{\mathrm{tt}}(\theta) \leqslant \mathsf{K} \leqslant \mathsf{H}^{\mathrm{tt}}(\theta)$ *in* $\mathrm{tilt}(\mathsf{H})$.

Proof. Write $\mathsf{H} = \mathsf{T} * \mathsf{F}$ for the torsion pair associated to K, and note that if $\theta \in \mathbf{C}(\mathsf{K})$ then the definitions imply $\mathsf{T} \subseteq \mathsf{H}^{\mathrm{tr}}(\theta)$ and $\mathsf{F} \subseteq \mathsf{H}^{\mathrm{tf}}(\theta)$, which under the bijections (2) gives us $\mathsf{H}_{\mathrm{tt}}(\theta) \leqslant \mathsf{K} \leqslant \mathsf{H}^{\mathrm{tt}}(\theta)$. Conversely if K lies in the said interval, then we have $\mathsf{T} \subseteq \mathsf{H}^{\mathrm{tr}}(\theta)$ and $\mathsf{F} \subseteq \mathsf{H}^{\mathrm{tf}}(\theta)$, and hence $\mathbf{C}(\mathsf{H}^{\mathrm{tr}}\theta) \subseteq \mathbf{C}(\mathsf{T})$ and $\mathbf{C}(\mathsf{H}^{\mathrm{tf}}\theta) \subseteq \mathbf{C}(\mathsf{F})$. This gives us the desired conclusion

$$\theta \quad \in \quad \mathbf{C}(\mathsf{H}^{\mathrm{tf}}\theta) \cap \left(-\mathbf{C}(\mathsf{H}^{\mathrm{tr}}\theta)\right) \quad \subseteq \quad \mathbf{C}\,\mathsf{F} \cap (-\mathbf{C}\,\mathsf{T}) = \mathbf{C}\,\mathsf{K}. \qquad \square$$

Given the above lemma we can now describe the interval $[\mathsf{H}_{\mathrm{tt}}\theta, \mathsf{H}^{\mathrm{tr}}\theta]$ in $\mathrm{tilt}(\mathsf{H})$ using semistable objects following [AP22]. Recall that given an Abelian category K and a parameter $\theta \in \mathrm{Hom}(\mathbf{K}\,\mathsf{K}, \mathbb{R})$, King [Kin94] defines the full subcategory of θ-*semistable objects in* K as

$$\mathsf{K}_{\mathrm{ss}}(\theta) = \{\mathsf{k} \in \mathsf{K} \mid \theta[\mathsf{k}] = 0, \quad \theta[\mathsf{s}] \geqslant 0 \text{ for all sub-objects } \mathsf{s} \hookrightarrow \mathsf{k}\}.$$

This is a wide (in particular, Abelian) subcategory of K. We say a θ-semistable object $\mathsf{k} \in \mathsf{K}$ is *stable* if it is simple in $\mathsf{K}_{\mathrm{ss}}(\theta)$, equivalently if the inequality $\theta[\mathsf{s}] > 0$ is strict for every sub-object $\mathsf{s} \hookrightarrow \mathsf{k}$ in K. If K is algebraic, then every θ-semistable object admits a filtration by θ-stable ones and thus $\mathsf{K}_{\mathrm{ss}}(\theta)$ is the extension closure of the set of θ-stable objects.

In the context of our algebraic Abelian category H with $\theta \in \Theta$, we clearly have $\mathsf{H}_{\mathrm{ss}}(\theta) = \mathsf{H}^{\mathrm{tr}}(\theta) \cap \mathsf{H}^{\mathrm{tf}}(\theta)$, and hence there is a decomposition

$$\mathsf{H} = \underbrace{\mathsf{H}_{\mathrm{tr}}(\theta) * \overbrace{\mathsf{H}_{\mathrm{ss}}(\theta)}^{}}_{\mathsf{H}^{\mathrm{tr}}(\theta)}\!\!\overbrace{\phantom{\mathsf{H}_{\mathrm{ss}}(\theta)} * \mathsf{H}_{\mathrm{tf}}(\theta)}^{\mathsf{H}^{\mathrm{tf}}(\theta)}.$$

In particular if $\mathsf{H}_{\mathrm{ss}}(\theta) = \mathsf{U} * \mathsf{V}$ is a torsion pair, then there is an induced torsion pair $\mathsf{H} = (\mathsf{H}_{\mathrm{tr}}\theta * \mathsf{U}) * (\mathsf{V} * \mathsf{H}_{\mathrm{tf}}\theta)$ and the corresponding tilt $\mathsf{K} = \mathsf{V} * \mathsf{H}_{\mathrm{tf}}\theta * \mathsf{H}_{\mathrm{tr}}\theta[-1] * \mathsf{U}[-1]$ evidently lies in the interval $[\mathsf{H}_{\mathrm{tt}}\theta, \mathsf{H}^{\mathrm{tt}}\theta]$. Further considering θ as a stability parameter on K (using the isomorphism $\mathbf{K}\,\mathsf{K} \cong \mathbf{K}\,\mathsf{H}$), lemma 2.12 shows θ is non-negative on every class of K so one computes

$$\mathsf{K}_{\mathrm{ss}}(\theta) = \{\mathsf{k} \in \mathsf{K} \mid \theta[\mathsf{k}] = 0\} \quad = \mathsf{V} * \mathsf{U}[-1]$$

and this clearly lies in $\mathrm{tilt}(\mathsf{H}_{\mathrm{ss}}\theta)$. The following theorem shows that considering semistable objects thus gives a poset isomorphism $[\mathsf{H}_{\mathrm{tt}}\theta, \mathsf{H}^{\mathrm{tt}}\theta] \to \mathrm{tilt}(\mathsf{H}_{\mathrm{ss}}\theta)$.

Lemma 2.13. *The maps below are well defined and give mutually inverse poset isomorphisms*

$$\mathrm{tilt}\,(\mathsf{H}_{\mathrm{ss}}\theta) \underset{\langle\,(\,\text{-}\,), \mathsf{H}_{\mathrm{tf}}\theta, \mathsf{H}_{\mathrm{tr}}\theta[-1]\,\rangle}{\overset{(\,\text{-}\,)_{\mathrm{ss}}\theta}{\longleftrightarrow}} \left[\mathsf{H}_{\mathrm{tt}}\theta, \mathsf{H}^{\mathrm{tt}}\theta\right] \quad \subseteq \mathrm{tilt}(\mathsf{H})$$

Further the covering relations in $\mathrm{tilt}(\mathsf{H}_{\mathrm{ss}}\theta)$ *are precisely those that remain covering in* $\mathrm{tilt}(\mathsf{H})$ *under the above correspondence, and the brick-labels coming from the two posets coincide.*

Proof. This is essentially [AP22, theorem 1.4], which is stated in terms of torsion classes and gives the poset isomorphisms

$$\mathrm{tors}\,(\mathsf{H}_{\mathrm{ss}}\theta) \underset{\mathsf{H}_{\mathrm{tr}}(\theta) * (\,\text{-}\,)}{\overset{(\,\text{-}\,) \cap \mathsf{H}_{\mathrm{ss}}(\theta)}{\longleftrightarrow}} \left[\mathsf{H}_{\mathrm{tt}}\theta, \mathsf{H}^{\mathrm{tt}}\theta\right] \quad \subseteq \mathrm{tors}(\mathsf{H}). \qquad \square$$

A consequence of lemmas 2.12 and 2.13 is that in addition to being a tilt of H, every intermediate heart K with $\theta \in \mathbf{C}(\mathsf{K})$ is also a tilt of $\mathsf{H}^{\mathrm{tt}}(\theta)$. Indeed, we have $\mathsf{K} = (\mathsf{V} * \mathsf{H}_{\mathrm{tf}}\theta) * (\mathsf{H}_{\mathrm{tr}}\theta * \mathsf{U})[-1]$ for some torsion pair $\mathsf{H}_{\mathrm{ss}}(\theta) = \mathsf{U} * \mathsf{V}$, and we can verify that U is a torsion class in $\mathsf{H}^{\mathrm{tt}}(\theta)$.

Further, lemma 2.13 gives us the following characterisations of $\mathsf{H}^{\mathrm{tt}}(\theta)$.

Corollary 2.14. *Given an intermediate heart* $\mathsf{K} \in \operatorname{tilt}(\mathsf{H})$ *and a vector* $\theta \in \mathbf{C}\,\mathsf{K}$*, the following statements are equivalent.*

(1) The heart K *is maximal among tilts of* H *with* θ *in their heart cone, i.e.* $\mathsf{K} = \mathsf{H}^{\mathrm{tt}}(\theta)$.

(2) The categories of θ*-semistable objects in* H *and* K *coincide, i.e.* $\mathsf{H}_{\mathrm{ss}}(\theta) = \mathsf{K}_{\mathrm{ss}}(\theta)$.

(3) The category $\mathsf{H}_{\mathrm{ss}}(\theta)$ *lies in* K.

(4) The category $\mathsf{K}_{\mathrm{ss}}(\theta)$ *lies in* H.

We say a vector θ is *generic* in a cone σ if it lies in σ but not in any proper face of σ, equivalently if it is contained in the interior of σ in the Euclidean space $\operatorname{span}(\sigma) \subset \Theta$. If θ, θ' are both generic in some cone $\sigma \in \operatorname{HFan}(\mathsf{H})$, it is easy to see that all their associated numerical categories coincide i.e. $\mathsf{H}_{\mathrm{tr}}(\theta) = \mathsf{H}_{\mathrm{tr}}(\theta')$, $\mathsf{H}_{\mathrm{tf}}(\theta) = \mathsf{H}_{\mathrm{tf}}(\theta')$, and so forth. Thus we use the notations $\mathsf{H}_{\mathrm{tr}}(\sigma)$, $\mathsf{H}^{\mathrm{tr}}(\sigma)$, $\mathsf{H}_{\mathrm{tf}}(\sigma)$, $\mathsf{H}^{\mathrm{tf}}(\sigma)$, $\mathsf{H}_{\mathrm{tt}}(\sigma)$, $\mathsf{H}^{\mathrm{tt}}(\sigma)$, and $\mathsf{H}_{\mathrm{ss}}(\sigma)$ to denote the respective categories associated to any generic vector $\theta \in \sigma$.

Remark 2.15 (Zooming into the heart fan). Given a stability parameter $\theta \in \Theta$, the correspondence 2.13 can be used to essentially 'read off' the heart fan of $\mathsf{H}_{\mathrm{ss}}(\theta)$ from that of H. To see this, note that the exact inclusion $\mathsf{H}_{\mathrm{ss}}(\theta) \hookrightarrow \mathsf{H}$ induces a map of Grothendieck groups which can be factored through its image $\mathfrak{h}_{\mathrm{ss}}$ as

$$\mathbf{K}\,\mathsf{H}_{\mathrm{ss}} \twoheadrightarrow \mathfrak{h}_{\mathrm{ss}} \hookrightarrow \mathfrak{h} \xrightarrow{\sim} \mathbf{K}\,\mathsf{H},$$

where $\mathfrak{h}_{\mathrm{ss}}$ is again a free Abelian group of finite rank. The heart fan of $\mathsf{H}_{\mathrm{ss}}(\theta)$ with respect to $\mathfrak{h}_{\mathrm{ss}}$ thus lives in the vector space $\Theta_{\mathrm{ss}} = \operatorname{Hom}_{\mathbb{Z}}(\mathfrak{h}_{\mathrm{ss}}, \mathbb{R})$ which is naturally a quotient of Θ. It can be shown that the surjection $\Theta \twoheadrightarrow \Theta_{\mathrm{ss}}$ is such that the image of (the sub-fan generated by)

$$\{\sigma \in \operatorname{HFan}(\mathsf{H}) \mid \theta \text{ lies in some face of } \sigma\}$$

is a sub-fan of $\operatorname{HFan}(\mathsf{H}_{\mathrm{ss}}\theta)$. More precisely, given a heart $\mathsf{K} \in [\mathsf{H}_{\mathrm{tt}}\theta, \mathsf{H}^{\mathrm{tt}}\theta]$ and the corresponding category $\mathsf{K}_{\mathrm{ss}}(\theta) \in \operatorname{tilt}(\mathsf{H}_{\mathrm{ss}}\theta)$, it can be shown that the image of $\mathbf{C}(\mathsf{K})$ is always a face of $\mathbf{C}(\mathsf{K}_{\mathrm{ss}}\theta)$ and is in fact equal to $\mathbf{C}(\mathsf{K}_{\mathrm{ss}}\theta)$ if K is algebraic.

To see an example where the image of $\mathbf{C}(\mathsf{K})$ is a proper face of $\mathbf{C}(\mathsf{K}_{\mathrm{ss}})$, one should examine the category $\mathsf{H} = \operatorname{Rep}(2 \rightrightarrows 1)$ associated to the 2-Kronecker quiver [BPPW24, example 4.6], and the stability parameter $\theta = (-1, 1)$ given in the basis dual to vertex simples.

§ 3 Algebraic hearts on flopping contractions

As in the introduction, let $\mathsf{Z} = \operatorname{Spec} \mathsf{R}$ be a complete local isolated compound du Val (cDV) singularity, equivalently the spectrum of a complete local Gorenstein $\mathbb{C}$-algebra R such that the unique singularity at the maximal ideal $\mathfrak{m}$ is at worst terminal. In particular, R is a normal domain.

A *flopping contraction* over Z [KM98, definition 6.10] is a projective birational morphism $\pi : \mathsf{X} \to \mathsf{Z}$ such that X is normal and the exceptional locus is codimension 2. In particular, canonical divisor on X is numerically π-trivial (equivalently $\pi^*\omega_{\mathsf{Z}} = \omega_{\mathsf{X}}$, i.e. π is *crepant*). By Kawamata's vanishing theorem, the map π is acyclic i.e. $\mathbf{R}\pi_*\mathcal{O}_{\mathsf{X}} = \mathcal{O}_{\mathsf{Z}}$ and thus the exceptional fiber $\underline{\mathsf{C}} = \pi^{-1}[\mathfrak{m}]$ is such that the reduced subscheme $\mathsf{C} = \underline{\mathsf{C}}_{\mathrm{red}}$ is a finite collection of rational curves [see e.g. Van04, lemma 3.4.1].

The integral components of C can be naturally indexed over some Dynkin data by considering a generic hyperplane section (*general elephant*) $\overline{\mathsf{Z}} = \operatorname{Spec} \overline{\mathsf{R}} \hookrightarrow \operatorname{Spec} \mathsf{R}$ and the pullback $\overline{\mathsf{X}} = \mathsf{X} \times_{\mathsf{Z}} \overline{\mathsf{Z}}$. Indeed by assumptions on R, the variety $\overline{\mathsf{Z}}$ is the germ of a canonical surface singularity and the map $\overline{\mathsf{X}} \to \overline{\mathsf{Z}}$ is a crepant partial resolution which is independent of the choice of general elephant [Rei, theorem 1.14]. In particular $\overline{X}$ is obtained by contracting a subset of exceptional curves in the minimal resolution of $\overline{Z}$. By the McKay correspondence [McK80], the exceptional curves in the minimal resolution are indexed over some ADE Dynkin graph Δ. Recording the indices of the contracted curves in a subset $\mathfrak{J} \subset \Delta$ shows that the integral components of C are naturally indexed over the Dynkin subgraph $\Delta \setminus \mathfrak{J}$ and we can write the reduced exceptional fiber as $C = \bigcup_{i \in \Delta \setminus \mathfrak{J}} C_i$ where each component C_i is a $\mathbb{P}^1$.

It is convenient to view Δ as sitting inside the associated extended (affine) Dynkin diagram $\underline{\Delta}$ with extended vertex $0 \in \underline{\Delta} \setminus \Delta$. This data $\mathfrak{J} \subset \Delta \subset \underline{\Delta}$ controls much of the homological algebra of R as we now describe.

For each exceptional curve C_i $(i \in \Delta \setminus \mathfrak{J})$, Van den Bergh [Van04, §3.5] gives the construction of a distinguished vector bundle $\mathcal{N}_i$ on X. This is such that if we write $\mathcal{N}_0 = \mathcal{O}_X$, the resulting bundle $\mathcal{V}(\frac{X}{Z}) = \bigoplus_{i \in \underline{\Delta} \setminus \mathfrak{J}} \mathcal{N}_i$ is tilting. Consequently X is derived equivalent to the basic R algebra $\Lambda = \mathrm{End}_X \mathcal{V}(\frac{X}{Z})$ via the functor

$$\mathrm{VdB} : \mathbf{D}^b\Lambda \xrightarrow{(-)\otimes^{\mathbf{L}}_{\Lambda} \mathcal{V}(\frac{X}{Z})} \mathbf{D}^b X. \tag{4}$$

This equivalence maps the natural heart $\mathit{mod}\Lambda \subset \mathbf{D}^b\Lambda$ to the *category of perverse sheaves* $\mathcal{P}er(\frac{X}{Z}) = \mathrm{VdB}(\mathit{mod}\Lambda)$, which should be thought of as the 'standard' heart in $\mathbf{D}^b X$ owing to its amiable cohomological properties. By [Van04, corollary 3.2.8],

$$\mathcal{P}er(\tfrac{X}{Z}) = \left\{ x \in \mathcal{C}oh\,X\,[0,1] \;\middle|\; \begin{array}{c} \mathbf{R}^1\pi_*(H^0 x) = 0, \quad \pi_*(H^{-1}x) = 0, \\ \mathrm{Hom}(c, H^{-1}x) = 0 \text{ whenever } c \in \mathcal{C}oh\,X \text{ satisfies } \mathbf{R}\pi_* c = 0 \end{array} \right\}. \tag{5}$$

Indecomposable projective Λ-modules are naturally in bijection with the summands $\mathcal{N}_i \subset \mathcal{V}(\frac{X}{Z})$ (and hence with the vertices of $\underline{\Delta} \setminus \mathfrak{J}$), where the summand $\mathcal{N}_i$ determines the indecomposable projective Λ-module

$$P_i = \mathrm{Hom}_X\left(\mathcal{V}(\tfrac{X}{Z}), \mathcal{N}_i\right).$$

By [Van04, proposition 3.5.7], the simple Λ-module S_i dual to P_i can be tracked across (4) as

$$\mathrm{VdB}(\; S_i \;) = \begin{cases} \omega_{\underline{C}}[1], & i = 0 \\ \mathcal{O}_{C_i}(-1), & i \in \Delta \setminus \mathfrak{J} \end{cases}.$$

Now under the equivalence (4), the subcategory $\mathbf{D}^0 X \subset \mathbf{D}^b X$ of complexes with cohomology supported within C is identified with $\mathbf{D}^{\mathrm{fl}}\Lambda \subset \mathbf{D}^b\Lambda$, the thick subcategory generated by the simples $\{S_i \mid i \in \underline{\Delta} \setminus \mathfrak{J}\}$ (equivalently, the subcategory of complexes whose cohomology modules have finite length over Λ).

We write $\mathit{per}(\frac{X}{Z}) = \mathcal{P}er(\frac{X}{Z}) \cap \mathbf{D}^0 X$ for the subcategory which then corresponds to the natural heart $\mathit{flmod}\Lambda \subset \mathbf{D}^{\mathrm{fl}}\Lambda$ of finite length Λ-modules. Evidently, this is an algebraic Abelian category with simple objects $\{S_i \mid i \in \underline{\Delta} \setminus \mathfrak{J}\}$.

Thus we seek to examine t-structures on $\mathbf{D}^0 X$ intermediate with respect to $\mathit{per}(\frac{X}{Z})$, equivalently the torsion theories on $\mathit{flmod}\Lambda$. In this section we give a complete description of the algebraic tilts and their partial order.

§ 3.1 Sets with mutation. A recurring motif in our treatment of algebraic hearts is that of mutation combinatorics, which allow us to enumerate sets over Dynkin data. We briefly explain the construction in an abstract setting, and recognise its various manifestations as they come up in the rest of the section.

Let G be a Dynkin graph with associated Weyl group $W(G)$, which is generated by simple reflections $\{s_i \mid i \in G\}$. When $W(G)$ is finite there is a unique longest element w_G in the weak (Bruhat) order, and further [e.g. by IW, lemma 1.2] there is an involution $\mathrm{inv}_G : G \to G$ such that $w_G s_i w_G = s_{\mathrm{inv}_G(i)}$. If $W(G)$ is not finite, we simply declare inv_G to be the identity.

Then for any subgraph $J \subset G$, this defines a map $\iota_J : G \setminus J \to G$ given by $\iota_J(i) = \mathrm{inv}_{J+i}(i)$ where inv_{J+i} is the involution for the full subgraph $J \cup \{i\}$.

We then say the *simple mutation of the Dynkin data* $J \subset G$ at $i \in G \setminus J$ is given by

$$\nu_i J = J \cup \{i\} \setminus \{\iota_J(i)\} \quad \subset G.$$

When iterating simple mutations, we omit brackets thus writing $\nu_{i_n}...\nu_{i_1}J$ to mean $\nu_{i_n}(\nu_{i_{n-1}}...\nu_{i_1}J)$, noting that the sequence makes sense only if $i_1 \in G \setminus J$ and $i_{j+1} \in G \setminus (\nu_{i_j}...\nu_{i_1}J)$ for each $j = 1, ..., n-1$. In this case we say the sequence of symbols $\nu = \nu_{i_n}...\nu_{i_1}$ is a J-*path of length* n.

Paths are composed in the obvious way, namely if ν is a J-path of length n and μ is a νJ-path of length m then the concatenation $\mu\nu$ is a J-path of length $m+n$. We treat the empty word $(\emptyset)$ as a J-path of length 0.

Clearly, simple mutation is involutive in the sense that $\nu_{\iota_J(i)}\nu_i J = J$. Given a J-path $\nu = \nu_{i_n}...\nu_{i_1}$, we can thus define a *reversed* J-*path* to be the νJ-path given by $\overline{\nu} = \nu_{\iota(i_1)}...\nu_{\iota(i_n)}$, where the maps ι in order correspond to the subsets J, $\nu_{i_1}J$, ... , $\nu_{i_{n-1}}...\nu_{i_1}J$.

When the choice of J is clear, we simply write ι instead of ι_J.

Definition 3.1. A *set with* G-*mutation* is a set A equipped with a map $\mathbb{J} : A \to 2^G$ that assigns each element of A to some subgraph of G, and a collection of functions $(\nu_i : \{a \in A \mid i \notin \mathbb{J}(a)\} \to A)_{i\in G}$ (called *simple mutations*) such that for each $a \in A$, we have $\mathbb{J}(\nu_i a) = \nu_i \mathbb{J}(a)$ and $\nu_{\iota(i)}(\nu_i a) = a$.

If A, A' are two sets with G-mutation, we say a map $f : A \to A'$ *respects mutation* if for all $a \in A$ and $i \in G \setminus \mathbb{J}(a)$, we have $\mathbb{J}(fa) = \mathbb{J}(a)$ and $\nu_i(fa) = f(\nu_i a)$.

Example 3.2. Given a Dynkin graph G, the set of all its subgraphs 2^G is naturally a set with G-mutation. This naturally breaks up into smaller sets with G-mutation called *mutation classes*, where the mutation class of $J \subseteq G$ is the subset of 2^G containing subgraphs that can be obtained from J by iterated mutation. Thus $\{\emptyset\}$ and $\{G\}$ are mutation classes of the trivial subgraphs $\emptyset$ and G respectively. Iyama–Wemyss [IW, §4.2] compute all mutation classes containing 'large' subgraphs of affine Dynkin diagrams, i.e. subgraphs whose complements contain 3 vertices. We draw the mutation class $\mathcal{E}_{7,4}$ below.

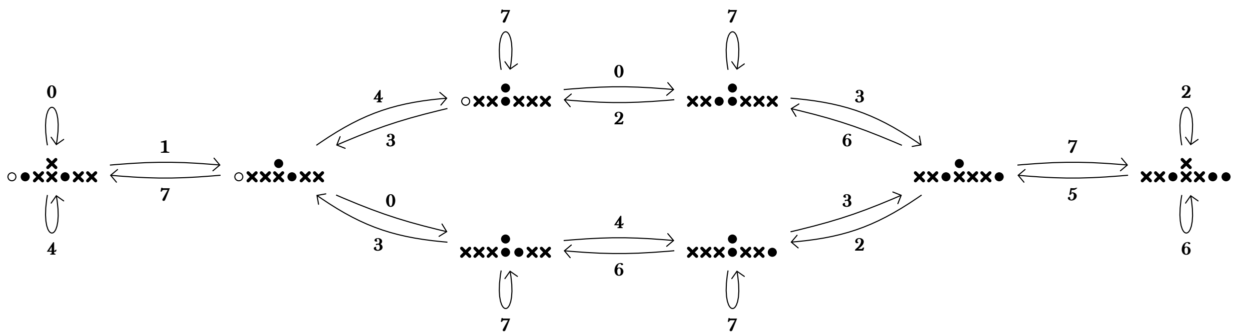
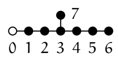


Figure 4. The mutation class of $J = \{2, 3, 5, 6, 7\}$ inside the $\widetilde{E}_7$ Dynkin graph , where subgraphs are indicated by marking off their vertices with a cross (✕). Arrows indicate simple mutation, the symbol ν has been omitted for brevity.

Example 3.3. If A is a set with an action of the Weyl group $W(G)$, then there is a natural G-mutation structure given by $\mathbb{J}(a) = \emptyset$, $\nu_i(a) = a \cdot s_i$ for all $a \in A$, $i \in G$. Thus for instance the set of chambers $\mathrm{Cham}(G)$ in the Tits cone of $W(G)$ is a set with G-mutation, where the (right)-action is by simple wall crossings.

More generally, a subset $J \subset G$ determines a hyperplane arrangement in Euclidean space [see for example IW, chapter 1], and any set with an action of the associated Deligne groupoid (also called the J-*cone groupoid*, see § 2.3.1 *ibid.*) is a set with G-mutation. This includes the prototypical example $\mathrm{Cham}(G, J)$, which is the set of chambers in the Tits cone of the hyperplane arrangement.

When $G = \Delta, \underline{\Delta}$ and $J = \mathfrak{J}$, we explain the construction of the hyperplane arrangement and the set $\mathrm{Cham}(G, J)$ in §§ 4.4 and 4.5. Other structures associated to the data $\mathfrak{J} \subset \Delta \subset \underline{\Delta}$, for instance the set of modifying R-modules (§ 3.3), the set of torsion(-free) classes in $\mathit{fl mod}\,\Lambda$ (§ 3.4), and the set of birational models of X (§ 5.1) are all shown to have a natural mutation structure which is related to that on the set of chambers.

Paths and exchange quivers. If A is a set with G-mutation, the *exchange quiver* $\mathrm{ExQuiv}(A)$ is a quiver with vertices A and a labelled arrow $i : a \to \nu_i a$ for each $a \in A$ and $i \in G \setminus \mathbb{J}(a)$. Paths in the exchange quiver can be described by a sequence of valid mutations from a specified starting vertex– it is easy to see that each $\mathbb{J}(a)$-path $\nu = \nu_{i_n} \dots \nu_{i_1}$ describes a unique path in the exchange quiver

$$a \xrightarrow{i_1} \nu_{i_1} a \xrightarrow{i_2} \dots \xrightarrow{i_n} \nu_{i_n} \nu_{i_{n-1}} \dots \nu_{i_1} a \tag{6}$$

which we call the *positive path* ν from a, and further every path in the exchange quiver corresponds to a unique pair (a, ν) in this way. It is convenient to write νa for the end-point of this path. The reversed path $\overline{\nu}$ is then a positive path from νa and we have $\overline{\nu}(\nu a) = a$.

If the positive path (6) has minimal length among all paths in $\mathrm{ExQuiv}(A)$ from a to νa, we say it is *minimal*. Clearly all minimal positive paths from a to νa have the same length, and a positive path ν from a is minimal if and only if the reversed path $\overline{\nu}$ from νa is so. Further, if two vertices a, b lie in the same connected component of $\mathrm{ExQuiv}(A)$ then it is always possible to find a minimal path ν from a such that $b = \nu a$.

Remark 3.4. In what follows, we will work with the ADE Dynkin graph Δ and its affine counterpart $\underline{\Delta}$. Thus to avoid confusion when considering subsets $J \subset \Delta \subset \underline{\Delta}$, we use the term *spherical* J*-paths* for the paths corresponding to the ambient graph $G = \Delta$ and reserve the unqualified term J-path to mean paths for $G = \underline{\Delta}$. It can be seen that for such J, there is a natural bijective correspondence between spherical J-paths and J-paths in which the symbol ν_0 does not occur.

Constructing algebraic hearts via mutation (§§ 3.2 to 3.5)

§ 3.2 Modifying modules and Brenner–Butler theory. The approach we take to describing intermediate algebraic t-structures on $\mathcal{fl}\mathcal{mod}\,\Lambda$ is via silting theory. Iyama–Wemyss [IW; IW14] observe that the theory is in fact controlled by the combinatorics of certain reflexive R-modules, which they use to build a family of non-commutative algebras starting from Λ which are derived equivalent to X. We summarise the results in a form suitable for our purposes.

Definition 3.5. An R-module N is said to be *modifying* if it is basic, reflexive, and its endomorphism algebra $\mathrm{End}_R(N)$ is Cohen–Macaulay. We say N is an *modifying generator* if in addition it contains R as a direct summand (equivalently if it is Cohen–Macaulay).

The key example of a modifying generator for us is the module $\pi_* \mathcal{V}(\frac{X}{Z})$, which Cohen–Macaulay with a Cohen–Macaulay endomorphism algebra by [Van04, proposition 3.2.10].

For a modifying R-module N, we define the *mutation class* of N, written $\mathrm{MM}^N(R)$, to be the set of isomorphism classes of modifying R-modules that admit a two-term approximation by $\mathrm{add}(N)$ and further have the same number of indecomposable summands as N. We write $\mathrm{MMG}^N(R) \subset \mathrm{MM}^N(R)$ for the subset of modifying generators in the mutation class.

Note that M lies in $\mathrm{MM}^N(R)$ if and only if N lies in $\mathrm{MM}^M(R)$ [IW, corollary 9.29], and in this case we say M and N are modifying modules in the same mutation class. This defines an equivalence relation on the class of modifying R-modules. Each mutation class then furnishes a family of derived-equivalent algebras via the following Auslander–McKay type correspondence.

Theorem 3.6 [IW, theorem 9.25]. *If* M, N *are modifying* R*-modules in the same mutation class then* $\mathrm{Hom}_R(N, M)$ *is a (classical) reflexive tilting* $\mathrm{End}_R(N)$*-module, and further every reflexive tilting* $\mathrm{End}_R(N)$*-module arises in this way from a unique module in* $\mathrm{MM}^N(R)$.

There is a natural isomorphism $\mathrm{End}_{\mathrm{End}_R N}(\mathrm{Hom}_R(N, M)) \cong \mathrm{End}_R(M)$ given by reflexive equivalence [IW14, lemma 2.5]. Then as is standard, the tilting module $\mathrm{Hom}_R(N, M)$ in the above setup induces quasi-inverse derived equivalences

$$\mathbf{D}^b(\mathrm{End}_R N) \underset{\mathbf{R}\mathrm{Hom}(\mathrm{Hom}_R(N,M),-)}{\overset{(-)\otimes^{\mathbf{L}}\mathrm{Hom}_R(N,M)}{\rightleftarrows}} \mathbf{D}^b(\mathrm{End}_R M)\,. \tag{7}$$

The Brenner–Butler theorem [see e.g. AHK, chapter 4, remark 3.10] relates the standard hearts under the above equivalence. Namely, writing

$$\begin{aligned} T &= \{x \in \mathit{mod}\,\mathrm{End}_R N \mid \quad \mathrm{Ext}^1(\mathrm{Hom}_R(N,M),x) = 0\},\\ F &= \{x \in \mathit{mod}\,\mathrm{End}_R N \mid \quad \mathrm{Hom}(\mathrm{Hom}_R(N,M),x) = 0\},\\ U &= \{y \in \mathit{mod}\,\mathrm{End}_R M \mid \quad y \otimes \mathrm{Hom}_R(N,M) = 0\},\\ V &= \{y \in \mathit{mod}\,\mathrm{End}_R M \mid \quad \mathrm{Tor}_1(y,\mathrm{Hom}_R(N,M)) = 0\}, \end{aligned}$$

the theorem states that we have torsion pairs

$$\mathit{mod}\,\mathrm{End}_R N = T * F, \qquad \mathit{mod}\,\mathrm{End}_R M = U * V$$

and the equivalences (7) identify $T \xrightarrow{\sim} V$, $F[1] \xrightarrow{\sim} U$. In particular, the image of $\mathit{mod}\,\mathrm{End}_R N$ is a (negative) tilt of $\mathit{mod}\,\mathrm{End}_R M$. Equivalently, the image of $\mathit{mod}\,\mathrm{End}_R M[-1]$ is a tilt of $\mathit{mod}\,\mathrm{End}_R N$.

We observe that the above equivalences and torsion pairs restrict verbatim to the subcategories of complexes with finite length cohomology [see also SY13].

§ 3.3 Mutation of modifying modules. Iyama–Wemyss' correspondence above is complemented by the following result which enables us to enumerate the entire mutation class of a modifying R-module by iterated simple mutation from the starting seed.

Theorem 3.7. *Let N be a modifying R-module with an indecomposable direct summand $L \subset N$. Then there exists a unique (up to isomorphism) modifying module $M \in \mathrm{MM}^N(R)$ such that $M = (N/L) \oplus K$, where K is the kernel of a minimal right* $\mathrm{add}(N/L)$*-approximation of L. We say M is obtained from N by* simple mutation *at L, and the following statements hold.*

(1) Simple mutation is involutive, i.e. K is an indecomposable summand of M and the corresponding simple mutation is isomorphic to N.

(2) The module M is not isomorphic to N. Further, modules obtained from N by simple mutation at distinct summands are non-isomorphic.

(3) Moreover, every $M \in \mathrm{MM}^N(R)$ can be obtained from N by applying a finite sequence of simple mutations, i.e. there is a sequence of modifying modules $N = M^0, M^1, \ldots, M^n = M$ with indecomposable summands $L^i \subset M^i$ such that M^{i+1} is obtained from M^i by simple mutation at L^i.

(4) In the above statement, if M, N are both modifying generators then the sequence of simple mutations can be chosen such that no summand L^i is isomorphic to R, i.e. each M^i is also a modifying generator.

Proof. Since R is a complete local commutative Noetherian ring, any R-module admits a minimal right approximation by the additive closure of any other R-module. In particular the module $M = (N/L) \oplus K$ exists, and is unique up to isomorphism since minimal approximations are so. By [IW14, lemma 4.11], M is modifying and thus lies in $\mathrm{MM}^N(R)$. Since M, N then have the same number of indecomposable summands, it follows that K is indecomposable. The remaining properties are reliant on the singularity of R being isolated, which guarantees all mutations are Artinian. The statement (1) is [IW, corollary 9.28], while (2),(3), and (4) are consequences of corollary 9.31 *ibid.* □

Given the setup of our problem, we fix once and for all the choice of modifying generator $N = \pi_* \mathcal{V}(\frac{X}{Z})$, which has endomorphism algebra $\Lambda = \mathrm{End}_R(N) \cong \mathrm{End}_X \mathcal{V}(\frac{X}{Z})$. The indecomposable summands of N (equivalently, those of $\mathcal{V}(\frac{X}{Z})$) are in bijection with the integral exceptional curves in X by construction; the following lemma recovers this bijection.

Lemma 3.8. *The vector bundle $\mathcal{V}(\frac{X}{Z})$ has precisely one indecomposable summand (namely $\mathcal{N}_0$) isomorphic to $\mathcal{O}_X$. Further for any non-free indecomposable summand $\mathcal{N}_i \subset \mathcal{N}$ ($\imath \in \Delta \setminus \mathfrak{J}$), the closed subscheme $c_1(\mathcal{N}_i^\vee) \subset X$ intersects the curve C_i exactly once and is disjoint from C_j for $j \neq i$.*

Proof. This is implicit in the construction of $\mathcal{V}(\frac{X}{Z})$, see [Van04, §§ 3.4 and 3.5]. □

The indexing of exceptional curves over $\Delta \setminus \mathfrak{J}$ therefore furnishes a natural indexing of the summands of N over $\underline{\Delta} \setminus \mathfrak{J}$. We now have the necessary ingredients to give $\mathrm{MM}^N(R)$ the structure of a set with $\underline{\Delta}$-mutation.

Theorem 3.9 [IW]. *Each $M \in \mathrm{MM}^N(R)$ can be assigned a unique subset $\mathbb{J}(M) \subset \underline{\Delta}$ and a bijection between $\underline{\Delta} \setminus \mathbb{J}(M)$ and the indecomposable summands of M (written $i \mapsto M_i$) such that the following hold.*

(1) We have $\mathbb{J}(N) = \mathfrak{J}$ and $N_i = \pi_* \mathcal{N}_i$ for each $i \in \underline{\Delta} \setminus \mathfrak{J}$.

(2) If M' is obtained from M by simple mutation at M_i, then we have $\mathbb{J}(M') = \nu_i\, \mathbb{J}(M)$ and $M'_j = M_j$ for $j \neq \iota(i)$.

Writing $\nu_i M$ for the simple mutation of M at M_i, this gives $\mathrm{MM}^N(R)$ the structure of a set with $\underline{\Delta}$-mutation.

Theorem 3.7 implies that the exchange quiver $\mathrm{ExQuiv}(\mathrm{MM}^N R)$ is connected, with at most one arrow from any vertex to another, and no arrow from a vertex to itself.

In particular given any $M, M' \in \mathrm{MM}^N(R)$, there is a $\mathbb{J}(M)$-path ν such that $M' = \nu M$. Further if M and M' are modifying generators then we necessarily have $M_0 = M'_0 = R$ and the path ν above can be chosen to be spherical. Thus we have

$$\mathrm{MM}^N(R) = \{\nu N \mid \nu \text{ a } \mathfrak{J}\text{-path}\}, \qquad \mathrm{MMG}^N(R) = \{\nu N \mid \nu \text{ a spherical } \mathfrak{J}\text{-path}\}.$$

§ 3.4 The mutation functors. We now use the above description of $\mathrm{MM}^N(R)$ to enumerate a large class of torsion theories and intermediate hearts for H.

To set notation, given $\mathfrak{J}$-paths ν, μ we write ${}_\mu\Lambda_\nu = \mathrm{Hom}_R(\nu N, \mu N)$ and note that this is a reflexive tilting $\mathrm{End}_R(\nu N)$-module by theorem 3.6. We omit the empty path $\emptyset$ from the notation, so for example we have $\Lambda_\nu = \mathrm{Hom}_R(\nu N, N)$, ${}_\nu\Lambda = \mathrm{Hom}_R(N, \nu N)$. Happily, this is consistent with $\Lambda = \mathrm{Hom}_R(N, N)$.

Then each module $\nu N \in \mathrm{MM}^N(R)$ can be used to obtain two intermediate hearts in $\mathrm{tilt}(H)$. Indeed the constructions in (7) are symmetric in the input data so we obtain a tilting Λ-module ${}_\nu\Lambda$ and a tilting ${}_\nu\Lambda_\nu$-module Λ_ν, which in turn give two *mutation functors* $\mathbf{D}^{\mathrm{fl}}\, {}_\nu\Lambda_\nu \rightrightarrows \mathbf{D}^{\mathrm{fl}}\Lambda$ defined as

$$\Phi_\nu(\text{-}) = (\text{-}) \otimes^{\mathbf{L}} {}_\nu\Lambda, \qquad \Psi_\nu(\text{-}) = \mathbf{R}\mathrm{Hom}(\Lambda_\nu, \text{-}).$$

Accordingly, Brenner–Butler theory gives us torsion pairs

(8) $$\mathit{flmod}\,\Lambda = T_\nu * F_\nu = U_\nu * V_\nu,$$

$$\begin{aligned} T_\nu &= \{x \in \mathit{flmod}\,\Lambda \mid \mathrm{Ext}^1({}_\nu\Lambda, x) = 0\} \qquad & U_\nu &= \{x \in \mathit{flmod}\,\Lambda \mid x \otimes \Lambda_\nu = 0\} \\ F_\nu &= \{x \in \mathit{flmod}\,\Lambda \mid \mathrm{Hom}({}_\nu\Lambda, x) = 0\} \qquad & V_\nu &= \{x \in \mathit{flmod}\,\Lambda \mid \mathrm{Tor}_1(x, \Lambda_\nu) = 0\} \end{aligned}$$

such that the squares below commute.

$$\begin{array}{ccc} \mathbf{D}^{\mathrm{fl}}\,{}_\nu\Lambda_\nu & \xrightarrow[\sim]{\Phi_\nu} & \mathbf{D}^{\mathrm{fl}}\Lambda \\ \uparrow & & \uparrow \\ \mathit{flmod}\,{}_\nu\Lambda_\nu & \xrightarrow{\sim} & F_\nu[1] * T_\nu \end{array} \qquad \begin{array}{ccc} \mathbf{D}^{\mathrm{fl}}\,{}_\nu\Lambda_\nu & \xrightarrow[\sim]{\Psi_\nu} & \mathbf{D}^{\mathrm{fl}}\Lambda \\ \uparrow & & \uparrow \\ \mathit{flmod}\,{}_\nu\Lambda_\nu & \xrightarrow{\sim} & V_\nu * U_\nu[-1] \end{array}$$

Abusing notation to write H for the standard hearts $\mathit{flmod}\,\Lambda \subset \mathbf{D}^{\mathrm{fl}}\Lambda$ as well as $\mathit{flmod}\,{}_\nu\Lambda_\nu \subset \mathbf{D}^{\mathrm{fl}}\,{}_\nu\Lambda_\nu$, each $\mathfrak{J}$-path ν thus gives us hearts

(9) $$\Phi_\nu H[-1] = F_\nu * T_\nu[-1], \qquad \Psi_\nu H = V_\nu * U_\nu[-1].$$

which are both intermediate with respect to $\mathsf{H} = \mathcal{f\ell mod}\Lambda$.

Of course for the purpose of defining mutation functors there is nothing special about the choice of N and Λ, and we can define the mutation functors and torsion theories starting at any modifying module. In particular if we use the reversed path $\overline{\nu}$ to analogously define the torsion theories

$$\mathcal{f\ell mod}_\nu\Lambda_\nu = \mathsf{U}_{\overline{\nu}} * \mathsf{V}_{\overline{\nu}} = \mathsf{T}_{\overline{\nu}} * \mathsf{F}_{\overline{\nu}},$$

then $\Phi_\nu, \Psi_{\overline{\nu}}$ restrict to mutually inverse equivalences $\mathsf{U}_{\overline{\nu}} \leftrightarrows \mathsf{F}_\nu[1]$, $\mathsf{V}_{\overline{\nu}} \leftrightarrows \mathsf{T}_\nu$, while $\Psi_\nu, \Phi_{\overline{\nu}}$ restrict to equivalences $\mathsf{T}_{\overline{\nu}} \leftrightarrows \mathsf{V}_\nu$, $\mathsf{F}_{\overline{\nu}} \leftrightarrows \mathsf{U}_\nu[-1]$. Thus the isomorphism $\overline{\nu}\nu\mathsf{N} \cong \mathsf{N}$ in $\mathrm{MM}^{\mathsf{N}}\mathsf{R}$ manifests itself as the identity of functors

$$\Phi_\nu \circ \Psi_{\overline{\nu}} \;\cong\; \Psi_\nu \circ \Phi_{\overline{\nu}} \;\cong\; \mathrm{id}.$$

If ν has length 1, we call the corresponding mutation functors *simple*. To simplify notation, we write Φ_i for the functor Φ_{ν_i} corresponding to a simple mutation, likewise Ψ_i, U_i, V_i, T_i, F_i have the obvious meaning.

§ 3.5 Algebraic hearts constructed via mutation. Evidently, the hearts (9) are algebraic and so the torsion theories (8) are functorially finite. Thus we have sub-posets of ftors(H), ftorf(H), alg-tilt(H) given by

$$\mathrm{tors}^-(\mathsf{H}) = \{\mathsf{T}_\nu \mid \nu \text{ is a } \mathfrak{J}\text{-path}\}, \qquad \mathrm{torf}^-(\mathsf{H}) = \{\mathsf{F}_\nu \mid \nu \text{ is a } \mathfrak{J}\text{-path}\}, \qquad \mathrm{tilt}^-(\mathsf{H}) = \{\Phi_\nu\mathsf{H}[-1] \mid \nu \text{ is a } \mathfrak{J}\text{-path}\},$$
$$\mathrm{tors}^+(\mathsf{H}) = \{\mathsf{U}_\nu \mid \nu \text{ is a } \mathfrak{J}\text{-path}\}, \qquad \mathrm{torf}^+(\mathsf{H}) = \{\mathsf{V}_\nu \mid \nu \text{ is a } \mathfrak{J}\text{-path}\}, \qquad \mathrm{tilt}^+(\mathsf{H}) = \{\Psi_\nu\mathsf{H} \mid \nu \text{ is a } \mathfrak{J}\text{-path}\}.$$

Each of the above sets is naturally in bijection with $\mathrm{MM}^{\mathsf{N}}(\mathsf{R})$ and the bijection simply identifies representative $\mathfrak{J}$-paths. This is well-defined because the constructions of § 3.4 depend only on the end points of positive paths in $\mathrm{ExQuiv}(\mathrm{MM}^{\mathsf{N}}\mathsf{R})$. In other words, if ν, ν' are $\mathfrak{J}$-paths such that $\nu\mathsf{N} = \nu'\mathsf{N}$ in $\mathrm{MM}^{\mathsf{N}}(\mathsf{R})$, then we also have $\mathsf{T}_\nu = \mathsf{T}_{\nu'}$, $\mathsf{U}_\nu = \mathsf{U}_{\nu'}$, and so on.

This bijection also enhances the above subposets with the structure of a set with $\underline{\Delta}$-mutation, where for instance the structure on $\mathrm{tilt}^+(\mathsf{H})$ is given by writing $\mathbb{J}(\Psi_\nu\mathsf{H}) = \nu\mathfrak{J}$ and $\nu_i\,(\Psi_\nu\mathsf{H}) = \Psi_{\nu_i\nu}\mathsf{H}$ whenever $i \in \underline{\Delta} \setminus \nu\mathfrak{J}$.

Further these bijections are compatible with the natural bijections among these sets, i.e. the bijections coming from identifying $\mathfrak{J}$-paths coincide with the poset isomorphisms

$$(10) \qquad \begin{array}{ccc} (\mathrm{tors}^+\,\mathsf{H}, \subseteq)^{\mathrm{op}} & \rightleftarrows & (\mathrm{torf}^+\,\mathsf{H}, \subseteq) \\ & \nwarrow\!\searrow \quad \swarrow\!\nearrow & \\ & (\mathrm{tilt}^+\,\mathsf{H}, \leqslant) & \end{array} \qquad \begin{array}{ccc} (\mathrm{tors}^-\,\mathsf{H}, \subseteq)^{\mathrm{op}} & \rightleftarrows & (\mathrm{torf}^-\,\mathsf{H}, \subseteq) \\ & \nwarrow\!\searrow \quad \swarrow\!\nearrow & \\ & (\mathrm{tilt}^-\,\mathsf{H}, \leqslant) & \end{array}$$

obtained by restricting (2).

In what follows we analyse the structure of these sub-posets and how they sit in relation with other torsion theories, first by examining the covering relations via tilting theory, and then by understanding the global behaviour through convex geometric constructions.

§ 4 The partial order of algebraic tilts

Continuing to work with the notation of § 3, the aim of this section is to examine the local and global structure of the partial orders $\mathrm{tilt}^\pm(\mathsf{H}) \subseteq \text{alg-tilt}(\mathsf{H}) \subseteq \mathrm{tilt}(\mathsf{H})$, and to prove the following result.

Theorem 4.1. *The poset inclusions* $\mathrm{tilt}^\pm(\mathsf{H}) \subset \text{alg-tilt}(\mathsf{H}) \subset \mathrm{tilt}(\mathsf{H})$ *satisfy the following.*

1. *There is a disjoint union decomposition* $\text{alg-tilt}(\mathsf{H}) = \mathrm{tilt}^+(\mathsf{H}) \sqcup \mathrm{tilt}^-(\mathsf{H})$. *In particular, every algebraic heart that is intermediate with respect to* H *is of the form* $\Phi_\nu\mathsf{H}[-1]$ *or* $\Psi_\nu\mathsf{H}$ *for some* $\mathfrak{J}$*-path* ν.
2. *The sub-posets* $\mathrm{tilt}^+(\mathsf{H})$ *and* $\mathrm{tilt}^-(\mathsf{H})$ *are saturated in* $\mathrm{tilt}(\mathsf{H})$*, i.e. given* $\mathsf{K} \leqslant \mathsf{K}' \leqslant \mathsf{K}''$ *in* $\mathrm{tilt}(\mathsf{H})$ *such that* K, K'' *lie in* $\mathrm{tilt}^+(\mathsf{H})$ *(resp.* $\mathrm{tilt}^-(\mathsf{H})$*), then so does* K'.

(3) *Intervals in* $\mathrm{tilt}^+(H)$ *and* $\mathrm{tilt}^-(H)$ *are finite, i.e. if* K, K' *both lie in* $\mathrm{tilt}^+(H)$ *(resp.* $\mathrm{tilt}^-(H)$*) and satisfy* $K \leqslant K'$*, then the interval* $[K, K'] \subset \mathrm{tilt}(H)$ *contains finitely many elements.*

Proof. Deferred until § 4.8. □

Thus algebraic tilts of H all arise via mutation; this justifies the terminology 'mutations of $\mathcal{per}\left(\frac{X}{Z}\right)$' used (e.g. in the introduction) to refer to elements of alg-tilt(H).

Covering relations via tilting theory (§§ 4.1 to 4.3)

§ 4.1 The tilting order. The set reftilt(Λ) of reflexive tilting Λ-modules has a natural partial order [AI12, theorem 2.11] with a well-understood Hasse quiver, and we can pull this across Iyama–Wemyss' correspondence (theorem 3.6) to get a partial order on $\mathrm{MM}^N(R)$.

To describe the partial orders, recall that the bijection between $\mathrm{MM}^N(R)$ and reftilt(Λ) given by theorem 3.6 is precisely the map $\nu N \mapsto {}_\nu\Lambda$. Thus every element of reftilt(Λ) can be written as ${}_\nu\Lambda$ for some $\mathfrak{J}$-path ν. Then the order on reftilt(Λ) (and hence on $\mathrm{MM}^N(R)$) is given as

$$\nu N \geqslant \mu N \text{ in } \mathrm{MM}^N(R) \iff {}_\nu\Lambda \geqslant {}_\mu\Lambda \text{ in } \mathrm{reftilt}\,\Lambda \iff \mathrm{Ext}^1_\Lambda({}_\nu\Lambda, {}_\mu\Lambda) = 0. \tag{11}$$

The Hasse quiver of reftilt(Λ) is described in terms of mutation of tilting modules in indecomposable summands. Instead of giving the definition of mutation in this context, we note that combining [Kim24, theorem 4.3] and [IW, theorem 9.6] shows that the bijection $\mathrm{Hom}_R(N, -) : \mathrm{MM}^N(R) \to \mathrm{reftilt}(\Lambda)$ naturally induces a bijection of indecomposable summands, and is compatible with mutation on both sides i.e. for $M \in \mathrm{MM}^N(R)$ and $i \in \underline{\Delta} \setminus \mathbb{J}(M)$, the mutation of $\mathrm{Hom}_R(N, M)$ in the indecomposable summand $\mathrm{Hom}_R(N, M_i)$ is precisely $\mathrm{Hom}_R(N, \nu_i M)$.

By examining the corresponding result for reftilt(Λ), we now show that covering relations in the poset of modifying modules correspond precisely to simple mutations.

Lemma 4.2. *The poset* $\mathrm{MM}^N(R)$ *with the partial order induced from* reftilt(Λ) *has a connected Hasse quiver with maximal element* N*. For each* $M \in \mathrm{MM}^N(R)$ *and* $i \in \underline{\Delta} \setminus \mathbb{J}(M)$*, one of* $\nu_i M > M$ *or* $M > \nu_i M$ *must hold and furthermore all covering relations (i.e. edges in the Hasse quiver) arise in this way.*

Proof. It is clear that N is maximal since Λ is so in the tilting order. Likewise, two tilting Λ-modules are related by a simple mutation if and only if they are related by a covering relation in the partial order [see e.g. IW18, proposition 4.4(3)], and hence the same holds for $\mathrm{MM}^N(R)$.

It follows that doubling the arrows of the Hasse quiver (i.e. adding an arrow $M \leftarrow M'$ for every existing arrow $M \to M'$) yields precisely $\mathrm{ExQuiv}(\mathrm{MM}^N R)$. Since the exchange quiver is connected, so is the Hasse quiver. □

In fact, examining the proof of [IW, proposition 9.19] which eventually shows that the exchange graph is connected, we see that paths in the Hasse quiver can be made monotone in the following sense: if M is a modifying R-module in the mutation class of N, then there is a positive path $\nu = \nu_{i_n} \ldots \nu_{i_1}$ such that $M = \nu N$ and there is a chain of covering relations $N > \nu_{i_1} N > \ldots > \nu_{i_n} \ldots \nu_{i_1} N$. We say such positive paths are *atomic*. Thus atomic paths are precisely those positive paths which lie in the Hasse quiver (considered a subquiver of $\mathrm{ExQuiv}\left(\mathrm{MM}^N R\right)$ in the obvious way).

Note that if M is a module in $\mathrm{MM}^N(R)$, then the mutation classes $\mathrm{MM}^M(R)$ and $\mathrm{MM}^N(R)$ coincide as sets. However, the natural choices of partial orders on the two (induced by tilting orders of endomorphism algebras) are distinct, and the superscript distinguishes the posets by indicating the maximal element.

§ 4.2 Simple mutations give simple tilts. We will now prove a series of lemmas examining how tilting objects and the induced intermediate hearts behave under simple mutations [see also HW18, §5].

Recall that for any $M \in \mathrm{MM}^{N}(R)$, we write H for the natural heart $\mathit{flmod}(\mathrm{End}_R M)$ in $\mathbf{D}^{\mathrm{fl}}(\mathrm{End}_R M)$. Each indecomposable summand $M_i \subset M$ $(i \in \underline{\Delta} \setminus \mathbb{J}(M))$ then gives an indecomposable projective $(\mathrm{End}_R M)$-module $P_i = \mathrm{Hom}_R(M, M_i)$, and hence a dual simple module $S_i \in H$.

To establish the result betrayed by the title of this subsection, i.e. that simple mutations correspond precisely to simple tilts, it is necessary to track simple objects under mutation functors.

Lemma 4.3. *For* $i \in \underline{\Delta} \setminus \mathfrak{J}$ *and* $S_i \in \mathit{flmod}\Lambda$, *we have* $\Psi_{\iota(i)}(S_i) = \Phi_i^{-1}(S_i) = S_{\iota(i)}[-1] \in \mathbf{D}^{\mathrm{fl}}{}_{\nu_i}\Lambda_{\nu_i}$.

Proof. This is [HW18, lemma 5.3], the only change is how we are indexing the simples. □

We now show that the intermediate hearts $\Phi_i H[-1]$ and $\Psi_i H$ are precisely simple tilts corresponding to S_i.

Lemma 4.4. *For* $i \in \underline{\Delta} \setminus \mathfrak{J}$, *we have* $T_i = {}^{\perp}\{S_i\}, F_i = \langle S_i \rangle$ *and* $U_i = \langle S_i \rangle, V_i = \{S_i\}^{\perp}$.

Proof. Since ${}_{\nu_i}\Lambda$ is a mutation of the tilting module Λ at an indecomposable summand with $\Lambda \geqslant {}_{\nu_i}\Lambda$ in the tilting order, there is an exact sequence

$$0 \to \Lambda \to P' \to {}_{\nu_i}\Lambda \to 0$$

with $P' \in \mathrm{add}\{P_j \mid j \neq i\}$. Since $\mathrm{Hom}(P_j, S_i) = 0$ for $j \neq i$, the exact sequence gives us $\mathrm{Hom}({}_{\nu_i}\Lambda, S_i) = 0$ showing $S_i \in F_i$. On the other hand if $x \in {}^{\perp}\{S_i\}$ then there is a projective cover $P \to x \to 0$ with $P \in \mathrm{add}\{P_j \mid j \neq i\}$, so noting that ${}_{\nu_i}\Lambda$ has projective dimension $\leqslant 1$ we get an exact sequence $\mathrm{Ext}^1({}_{\nu_i}\Lambda, P) \to \mathrm{Ext}^1({}_{\nu_i}\Lambda, x) \to 0$. But $P \in \mathrm{add}({}_{\nu_i}\Lambda)$ and ${}_{\nu_i}\Lambda$ has no self-extensions, so in fact all the terms in the sequence must vanish and we have $x \in T_i$. Thus we have shown ${}^{\perp}\{S_i\} \subseteq T_i$ and $\langle S_i \rangle \subseteq F_i$, and it is clear that equality must hold.

To conclude, note that we have $U_i = \Psi_i(F_{\iota(i)})[1] = \Psi_i\langle S_{\iota(i)}\rangle[1] = \langle S_i \rangle$ where we use lemma 4.3 and that equivalences commute with extension closure. The equality $V_i = \{S_i\}^{\perp}$ follows. □

The final result in the subsection extends lemma 4.3 and examines the behaviour of other simples under simple mutations by tracking their **K**-theory classes.

Lemma 4.5. *For* $i \in \underline{\Delta} \setminus \mathfrak{J}$, *let the integers* $(b_j)_{j \in \underline{\Delta} \setminus (\mathfrak{J}+i)}$ *be such that* N_i *has minimal* $\mathrm{add}(N/N_i)$*-approximation*

$$0 \to N_i \to \bigoplus_{j \in \underline{\Delta} \setminus (\mathfrak{J}+i)} N_j^{\oplus b_j}.$$

Then for a simple $S_j \in \mathit{flmod}_{\nu_i}\Lambda_{\nu_i}$ *with* $j \neq \iota(i)$, *we have that* $\Phi_i(S_j)$ *and* $\Psi_i(S_j)$ *lie in the subcategory* $\langle S_i, S_j \rangle \subset \mathit{flmod}\Lambda$. *Further, in any Jordan–Hölder filtration of* $\Phi_i(S_j)$ *(resp.* $\Psi_i(S_j)$*), the simple* S_j *occurs exactly once while the simple* S_i *occurs* b_j *times.*

Proof. It is clear from lemma 4.4 that both $\Phi_i(S_j)$ and $\Psi_i(S_j)$ are modules, in particular $\Phi_i(S_j) = S_j \otimes {}_{\nu_i}\Lambda$ and $\Psi_i(S_j) = \mathrm{Hom}(\Lambda_{\nu_i}, S_j)$. The result then is [Wem18, lemma 5.7]. □

§ 4.3 Composing simple tilts. By comparing paths in Hasse quivers, we now establish that the mutation order is compatible with the standard order $(\leqslant)$ on $\mathrm{tilt}^{\pm}(H)$, equivalently the containment orders on $\mathrm{tors}^{\pm}(H)$ and $\mathrm{torf}^{\pm}(H)$. For this, it is necessary to understand how mutation functors compose.

Theorem 4.6. *Let* ν *be a positive path starting at* N, *and* $i \in \underline{\Delta} \setminus \nu\mathfrak{J}$. *Then the following are equivalent:*

(1) $\nu N > \nu_i \nu N$ *in* $\mathrm{MM}^{N}(R)$.

(2a) $T_\nu \supset T_{\nu_i \nu}$, *equivalently* $F_\nu \subset F_{\nu_i \nu}$.

(2b) $F_{\nu_i\nu} = \Phi_\nu F_i * F_\nu$, *equivalently* $T_{\nu_i\nu} = \Phi_\nu T_i \cap T_\nu$.

(2c) *In* $\mathcal{fl}mod_\nu\Lambda_\nu$ *we have* $S_i \in \Phi_\nu^{-1} T_\nu$.

(2d) *The heart* $\Phi_\nu(\Phi_i H)[-1] \subset \mathbf{D}^{fl}\Lambda$ *is intermediate with respect to* H, *i.e. lies in* $H[-1, 0]$.

(2e) *There is a natural isomorphism of functors* $\Phi_{\nu_i\nu} \cong \Phi_\nu \circ \Phi_i$.

(3a) $U_\nu \subset U_{\nu_i\nu}$, *equivalently* $V_\nu \supset V_{\nu_i\nu}$.

(3b) $U_{\nu_i\nu} = U_\nu * \Psi_\nu U_i$, *equivalently* $V_{\nu_i\nu} = V_\nu \cap \Psi_\nu V_i$

(3c) *In* $\mathcal{fl}mod_\nu\Lambda_\nu$ *we have* $S_i \in \Psi_\nu^{-1} V_\nu$.

(3d) *The heart* $\Psi_\nu(\Psi_i H) \subset \mathbf{D}^{fl}\Lambda$ *is intermediate with respect to* H.

(3e) *There is a natural isomorphism of functors* $\Psi_{\nu_i\nu} \cong \Psi_\nu \circ \Psi_i$.

(4) *There is an isomorphism of bimodules* ${}_{\nu_i\nu}\Lambda \cong {}_{\nu_i\nu}\Lambda_\nu \otimes^{\mathbf{L}} {}_\nu\Lambda$, *where the tensor product is over* ${}_\nu\Lambda_\nu$.

Before proving the above theorem, we note some immediate consequences.

Corollary 4.7. *If* ν *is a* $\mathfrak{J}$*-path and* $i \in \underline{\Delta} \setminus \nu\mathfrak{J}$ *is such that* $\nu N > \nu_i\nu N$, *then* $\mathrm{tilt}(H)$ *has covering relations*

$$\Phi_\nu H[-1] \lessdot \Phi_{\nu_i\nu} H[-1], \qquad \Psi_\nu H \gtrdot \Psi_{\nu_i\nu} H$$

with brick labels $\Phi_\nu S_i$ *and* $\Psi_\nu S_i$ *respectively. Further, every covering relation in* $\mathrm{tilt}(H)$ *which involves an element of* $\mathrm{tilt}^{\pm}(H)$ *is of the above form. Thus the bijections of* $\mathrm{MM}^N(R)$ *with* $\mathrm{tilt}^{\pm}(H)$ *induce an isomorphism of the Hasse quiver of* $(\mathrm{MM}^N R, \leqslant)$ *with that of* $(\mathrm{tilt}^+ H, \leqslant)$ *and that of* $(\mathrm{tilt}^- H, \leqslant)^{op}$.

Proof. That the relations written are covering with the given brick label is immediate.

Now if we have $K \lessdot \Psi_\nu H$ in $\mathrm{tilt}(H)$, then by theorem 2.4 the corresponding brick label b is a simple object of $\Psi_\nu H$ which lies in $H \cap \Psi_\nu H = V_\nu$ and K is the tilt of $\Psi_\nu H$ in the torsion class $\langle b \rangle$. In particular, we have $b = \Psi_\nu S_i$ for some $i \in \underline{\Delta} \setminus \mathfrak{J}$, so that theorem 4.6 ((3c)$\Rightarrow$(1)) gives us $\nu N > \nu_i\nu N$ and further $K = \Psi_{\nu_i\nu} H$.

An analogous reasoning shows covering relations of the form $K \gtrdot \Psi_\nu H$, $K \lessdot \Phi_\nu H[-1]$, and $K \gtrdot \Phi_\nu H[-1]$ are also of the given form. □

Remark 4.8. The above result as stated does *not* imply that the bijection $\mathrm{MM}^N(R) \to \mathrm{tilt}^+(H)$ is an isomorphism of posets. Indeed it is possible for non-isomorphic posets to have isomorphic Hasse quivers, for example compare the posets

$$\left\{ \pm\frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\} \subset (\mathbb{Q}, \leqslant), \qquad (\mathbb{N}, \leqslant) \sqcup (\mathbb{N}, \leqslant)^{op}.$$

Thus we need to eliminate the possibility of $\mathrm{Hasse}(\mathrm{MM}^N R)$ having infinite bounded intervals, which we do in § 4.8 by leveraging global structural results on the convex-geometric counterpart of this poset.

Corollary 4.9. *Given a positive path* $\nu = \nu_{i_n} \dots \nu_{i_1}$ *starting at* N, *one can write* $\Phi_\nu = \Upsilon_{i_1} \circ \Upsilon_{i_2} \circ \dots \circ \Upsilon_{i_n}$ *for some sequence of simple mutations* $\Upsilon_i \in \{\Phi_i, \Psi_i\}$. *Further if the path* ν *is atomic (i.e.* $N > \nu_{i_1} N > \nu_{i_2}\nu_{i_1} N > \dots > \nu N$*), then each* Υ_i *above is of the form* Φ_i *and we have* $\Phi_\nu = \Phi_{i_1} \circ \Phi_{i_2} \circ \dots \circ \Phi_{i_n}$.

The analogous statement holds for Ψ_ν.

Proof. For $n \geqslant 1$ we may write $\nu = \nu_i \nu'$ where $i = i_n$ and $\nu' = \nu_{i_{n-1}} \dots \nu_{i_1}$ is a shorter path. By lemma 4.2 we have either $\nu' N > \nu N$ or $\nu N > \nu' N$ in $\mathrm{MM}^N(R)$. In the first case theorem 4.6 gives us $\Phi_\nu = \Phi_{\nu'} \circ \Phi_i$, while in the latter case we write $\nu' = \nu_{\iota(i)}\nu$ to get $\Phi_\nu = \Phi_{\nu'} \circ (\Phi_{\iota(i)})^{-1} = \Phi_{\nu'} \circ \Psi_i$. The result follows by induction. □

We now prove theorem 4.6 in parts, showing (1) is equivalent to (2a)–(2e) and to (4). Proving the equivalence of (1) with (3a)–(3e) is entirely analogous. Note also that in (2a), (2b), (3a), and (3b), the equivalence of the two involved clauses follows from formal properties of torsion theories– for instance whenever $\mathsf{H} = \mathsf{T} * \mathsf{F} = \mathsf{T}' * \mathsf{F}'$ are two torsion pairs with $\mathsf{T} \subset \mathsf{T}'$, then $\mathsf{T}' \cap \mathsf{F}$ is a torsion class in the Abelian category $\mathsf{F} * \mathsf{T}[-1]$ with corresponding torsion-free class $\mathsf{F}' * \mathsf{T}[-1]$.

The statements (2e)⇒(2d) and (2b)⇒(2a) are immediate, while (4)⇒(2e) follows from the $\otimes^{\mathbf{L}}$–$\mathbf{R}\mathrm{Hom}$ adjunction [IR08, lemma 2.10]. Likewise the following are straightforward from the definitions of the involved torsion theories.

Proof of (2a)⇒(1). If (1) does not hold, then necessarily $\nu_i\nu\mathrm{N} > \nu\mathrm{N}$ by lemma 4.2 and hence there is a $y \in \operatorname{add}{}_{\nu}\Lambda$ and an exact sequence

$$0 \to {}_{\nu_i\nu}\Lambda \to y \to {}_{\nu}\Lambda \to 0.$$

In particular for $x \in \mathsf{T}_\nu$, we have $\mathrm{Ext}^1({}_\nu\Lambda, x) = 0$ and hence $\mathrm{Ext}^1(y, x) = 0$. Since $\mathrm{Ext}^2({}_\nu\Lambda, x) = 0$ for projective dimension reasons, the long exact sequence associated to $\mathrm{Hom}(-, x)$ shows $\mathrm{Ext}^1({}_{\nu_i\nu}\Lambda, x) = 0$ i.e. $x \in \mathsf{T}_{\nu_i\nu}$. Thus we have $\mathsf{T}_{\nu_i\nu} \supsetneq \mathsf{T}_\nu$, i.e. (2a) does not hold. □

Proof of (2e)⇒(2b). Using $\Phi_{\nu_i\nu} = \Phi_\nu \circ \Phi_i$, we have $\mathsf{T}_{\nu_i\nu} = \mathsf{H} \cap \Phi_{\nu_i\nu}\mathsf{H} = \mathsf{H} \cap \Phi_\nu(\Phi_i\mathsf{H}) = \mathsf{H} \cap (\Phi_\nu\mathsf{F}_i[1] * \Phi_\nu\mathsf{T}_i)$. But since $\Phi_\nu\mathsf{F}_i \subset \mathsf{H}[0,1]$, we have $\Phi_\nu\mathsf{F}_i[1] \cap \mathsf{H} = \{0\}$ and hence $\mathsf{T}_{\nu_i\nu} = \Phi_\nu\mathsf{T}_i \cap \mathsf{H}$. Likewise writing $\mathsf{H} = \mathsf{T}_\nu * \mathsf{F}_\nu$, we observe that $\Phi_\nu\mathsf{T}_i \subset \Phi_\nu\mathsf{H} = \mathsf{F}_\nu[1] * \mathsf{T}_\nu$, and hence $\Phi_\nu\mathsf{T}_i \cap \mathsf{F}_\nu = \{0\}$. Thus we have $\mathsf{T}_{\nu_i\nu} = \Phi_\nu\mathsf{T}_i \cap \mathsf{T}_\nu$ as required. □

Proof of (2d)⇒(2c). If (2c) does not hold, then noting that $\mathsf{H} = \Phi_\nu^{-1}\mathsf{F}_\nu[1] * \Phi_\nu^{-1}\mathsf{T}_\nu$ is a torsion theory on $\mathbf{D}^{\mathrm{fl}}\,{}_\nu\Lambda_\nu$, we have that $\mathsf{S}_i \in \Phi_\nu^{-1}\mathsf{F}_\nu[1]$ and hence $\Phi_\nu\mathsf{S}_i \in \mathsf{F}_\nu[1] \subset \mathsf{H}[1]$. But then we see that $\Phi_\nu(\Phi_i\mathsf{H})[-1] = \Phi_\nu\mathsf{F}_i * \Phi_\nu\mathsf{T}_i[-1]$ contains the object $\Phi_\nu\mathsf{S}_i$ (since $\mathsf{S}_i \in \mathsf{F}_i$) and hence cannot be intermediate. □

The statement (1)⇒(4) is proven in [HW18, theorem 4.6], we state the proof here for convenience.

Proof of (1)⇒(4). If we have $\nu\mathrm{N} > \nu_i\nu\mathrm{N}$, then by definition of the mutation order we have ${}_\nu\Lambda > {}_{\nu_i\nu}\Lambda$ as tilting Λ-modules. Then applying [HW18, proposition B.1] with $\mathrm{T} = {}_\nu\Lambda$ and $\Gamma = {}_\nu\Lambda_\nu$, we see that there is a Λ-module isomorphism ${}_{\nu_i\nu}\Lambda_\nu \otimes^{\mathbf{L}} {}_\nu\Lambda \cong {}_{\nu_i\nu}\Lambda$. To show this is a bimodule isomorphism, first note that ${}_{\nu_i\nu}\Lambda_\nu \otimes^{\mathbf{L}} {}_\nu\Lambda$ is concentrated in degree zero so we can replace the tensor product with its non-derived version. Then the given isomorphism of Λ-modules factors as

$${}_{\nu_i\nu}\Lambda_\nu \otimes {}_\nu\Lambda \xrightarrow{\sim} \mathrm{Hom}_\Lambda({}_\nu\Lambda, {}_{\nu_i\nu}\Lambda) \otimes {}_\nu\Lambda \xrightarrow{\sim} {}_{\nu_i\nu}\Lambda,$$

where the first is reflexive equivalence $\mathrm{Hom}_\Lambda(\mathrm{Hom}_R(\mathrm{N}, \nu\mathrm{N}), \mathrm{Hom}_R(\mathrm{N}, {}_{\nu_i\nu}\mathrm{N})) \cong \mathrm{Hom}_R(\nu\mathrm{N}, {}_{\nu_i\nu}\mathrm{N})$ and the second is the evaluation map. Both of these preserve the $({}_{\nu_i\nu}\Lambda_{\nu_i\nu})^{\mathrm{op}}$-module structure, hence we have an isomorphism of bimodules as required. □

Note that we have proven the chain of equivalences (1)⇒ (4)⇒ (2e)⇒ (2b)⇒ (2a)⇒ (1). Since we also have (2e)⇒(2d)⇒(2c), showing (2c)⇒(1) will finish the proof of theorem 4.6.

Proof of (2c)⇒(1). This is essentially [HW18, lemma 5.4]. If (1) does not hold, then by lemma 4.2 we must have $\nu_i\nu\mathrm{N} > \nu\mathrm{N} = \nu_{\iota(i)}\nu_i\nu\mathrm{N}$. Thus applying (1)⇒(2e) to this chain of mutations, we have $\Phi_\nu = \Phi_{\nu_i\nu} \circ \Phi_{\iota(i)}$ and hence by lemma 4.3, $\Phi_\nu(\mathsf{S}_i) = \Phi_{\nu_i\nu}(\mathsf{S}_{\iota(i)})[1]$. But noting $\Phi_{\nu_i\nu}(\mathsf{S}_{\iota(i)}) \in \Phi_{\nu_i\nu}\mathsf{H} \subset \mathsf{H}[0,1]$, we immediately have that $\Phi_\nu(\mathsf{S}_i) \notin \mathsf{H}$ and hence $\mathsf{S}_i \notin \Phi_\nu^{-1}\mathsf{T}_\nu$ as required. □

In subsequent sections, we shed more light on the structure of these posets by constructing the associated heart cones and relating the order to convex-geometric data.

Partial orders via Coxeter geometry (§§ 4.4 to 4.8)

§ 4.4 Restricted root systems of affine type. The $\mathbf{K}$-theoretic McKay correspondence for minimal surfaces [GV83] identifies the Grothendieck group of the category of coherent sheaves with the root lattice of an affine Dynkin diagram, and this identification is such that the action of spherical twist functors on $\mathbf{K}$-theory is precisely by reflections in the Weyl group. In the more general setting of a crepant partial resolution (in particular the flopping contraction $X \to Z$ we are concerned with), Iyama–Wemyss [IW] develop analogous combinatorics of *restricted root systems* to describe the $\mathbf{K}$-theory.

The affine root lattice. We recap the unrestricted (Coxeter) setting first, see [Kac80, §1] for a detailed introduction. Given the affine Dynkin diagram $\underline{\Delta}$ of rank n, the *root lattice* $\mathfrak{h} = \mathfrak{h}(\underline{\Delta})$ is a free $\mathbb{Z}$-module with basis given by the *simple roots* $\{\alpha_i \mid i \in \underline{\Delta}\}$. The Cartan matrix associated to $\underline{\Delta}$ gives a degenerate symmetric bilinear form $(-,-)$ on $\mathfrak{h}$, and we use these to define the *simple reflections* $s_i : \mathfrak{h} \to \mathfrak{h}$ $(i \in \underline{\Delta})$ given by

$$s_i(\alpha_j) = \alpha_j - (\alpha_i, \alpha_j)\alpha_i. \tag{12}$$

These define the *Weyl group* $W(\underline{\Delta}) = \langle s_i \mid i \in \underline{\Delta}\rangle \subset GL(\mathfrak{h})$, which is isomorphic to the Coxeter group associated to $\underline{\Delta}$ with standard generators $\{s_i \mid i \in \underline{\Delta}\}$. In fact each subset $J \subseteq \underline{\Delta}$ determines a parabolic subgroup $W(J) = \langle s_i \mid i \in J\rangle \subseteq W(\underline{\Delta})$, which is the Coxeter group associated to the full subgraph spanned by J. If $J \neq \underline{\Delta}$ then this is a finite Coxeter group, and hence has a unique Coxeter element w_J (defined to be the longest element in the Bruhat order).

We say the *set of real roots* $\mathrm{Root}(\underline{\Delta})$ is the union of $W(\underline{\Delta})$-orbits of the simple roots. We say a real root is *positive* if it can be expressed as a non-negative linear combination of the simple roots, and write $\mathrm{Root}^+(\underline{\Delta})$ for the set of positive real roots. The set $\mathrm{Root}^-(\underline{\Delta})$ of *negative real roots* is defined likewise. Every real root is either positive or negative, and the map $\alpha \mapsto -\alpha$ gives a bijection between $\mathrm{Root}^+(\underline{\Delta})$ and $\mathrm{Root}^-(\underline{\Delta})$. In fact for each real root α, there is a unique reflection s (i.e. an element of order 2) in $W(\underline{\Delta})$ such that $s(\alpha) = -\alpha$, and this gives a bijection between the set of positive real roots and the set of reflections in the Weyl group.

Vertices in the diagram $\underline{\Delta}$ can be Lie-theoretically assigned numerical labels $(\delta_i)_{i\in\underline{\Delta}}$ which are computed and given for each Dynkin type in [Kac80, table Z]. This is always a tuple of positive integers, and the 'extended' vertex $0 \in \underline{\Delta} \setminus \Delta$ is assigned the integer $\delta_0 = 1$. Then the vector $\delta = \sum_{i\in\underline{\Delta}} \delta_i\alpha_i$ is called the *primitive positive imaginary root* in $\mathfrak{h}$, and any non-zero multiple of δ is called an *imaginary root*. The *root system* of $\underline{\Delta}$ is then the set of all (real and imaginary) roots in $\mathfrak{h}$, and is preserved by the action of the Weyl group which acts transitively on $\mathrm{Root}(\underline{\Delta})$ and fixes δ.

Restricted roots. Suppose $\mathfrak{J} \subset \underline{\Delta}$ is a subset and $|\underline{\Delta}\setminus\mathfrak{J}| \geqslant 2$. This defines a decomposition $\mathfrak{h}(\underline{\Delta}) = \mathfrak{h}(\mathfrak{J}) \oplus \mathfrak{h}(\underline{\Delta}\setminus\mathfrak{J})$, where we write $\mathfrak{h}(\mathfrak{J})$ for the span of the simple roots $\{\alpha_i \mid i \in \mathfrak{J}\}$ (and define $\mathfrak{h}(\underline{\Delta}\setminus\mathfrak{J})$ likewise). We say $\mathfrak{h}(\underline{\Delta}\setminus\mathfrak{J})$ is the *restricted root lattice* associated to $\mathfrak{J}$, and the *restricted root system* is the image of the root system under the surjection $\mathfrak{h}(\underline{\Delta}) \twoheadrightarrow \mathfrak{h}(\underline{\Delta}\setminus\mathfrak{J})$. The notions of restricted (simple, positive, negative) real roots and restricted imaginary roots are defined likewise as the non-zero images of the corresponding objects in $\mathfrak{h}$. In particular there are $|\underline{\Delta}\setminus\mathfrak{J}|$ restricted simple roots $\{\alpha_i \mid i \in \underline{\Delta}\setminus\mathfrak{J}\}$, and a *primitive positive restricted imaginary root* $\delta_{\mathfrak{J}} = \sum_{i\in\underline{\Delta}\setminus\mathfrak{J}} \delta_i\alpha_i$. Writing $\mathrm{Root}(\underline{\Delta},\mathfrak{J})$ for the set of restricted real roots (and defining the subsets $\mathrm{Root}^{\pm}(\underline{\Delta},\mathfrak{J})$ of positive and negative restricted real roots accordingly), we thus see that the restricted root system is given as

$$\underbrace{\mathrm{Root}^+(\underline{\Delta},\mathfrak{J}) \sqcup \mathrm{Root}^-(\underline{\Delta},\mathfrak{J})}_{\mathrm{Root}(\underline{\Delta},\mathfrak{J})} \sqcup \{n\delta_{\mathfrak{J}} \mid n \in \mathbb{Z}\setminus\{0\}\}.$$

To obtain the analog of a Weyl-group action for restricted roots, it is necessary to consider simultaneously all restricted root lattices $\mathfrak{h}(\underline{\Delta}\setminus v\mathfrak{J})$ ranging over $\mathfrak{J}$-paths v. In the Coxeter setting, a simple reflection $s_i \in W(\underline{\Delta})$ was characterised by the property of being an involution such that $s_i(\alpha_j) - \alpha_j$ is equal to $-2\alpha_i$ if $j = i$, and is a non-negative multiple of α_i otherwise. Iyama–Wemyss suggest the following generalisation for restricted root systems.

Proposition 4.10 [NW23, lemmas 5.1 and 5.2]. *For $\mathfrak{I} \subset \underline{\Delta}$ and $i \in \underline{\Delta} \setminus \mathfrak{I}$, the linear map $\mathfrak{h} \to \mathfrak{h}$ given by the action of $w_{\mathfrak{I}} w_{\mathfrak{I}+i} \in W(\underline{\Delta})$ maps the subset $\mathfrak{h}(\nu_i \mathfrak{I})$ isomorphically onto $\mathfrak{h}(\mathfrak{I})$, thus inducing an isomorphism*

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\;w_{\mathfrak{I}} w_{\mathfrak{I}+i}\;} & \mathfrak{h} \\ \downarrow & & \downarrow \\ \mathfrak{h}(\underline{\Delta} \setminus \nu_i \mathfrak{I}) & \xrightarrow[\sim]{\;\varphi_i\;} & \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{I}) \end{array} \tag{13}$$

This map preserves the root systems, in particular acting on the simple roots and the primitive positive imaginary root as

$$\varphi_i(\delta_{\nu_i \mathfrak{I}}) = \delta_{\mathfrak{I}}, \qquad \varphi_i(\alpha_{\iota(i)}) = -\alpha_i, \qquad \varphi_i(\alpha_j) - \alpha_j \;\in\; \mathbb{Z}_{\geqslant 0} \cdot \alpha_i \quad (j \neq i).$$

Further, defining $\varphi_{\iota(i)} : \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{I}) \to \mathfrak{h}(\underline{\Delta} \setminus \nu_i \mathfrak{I})$ analogously, we have $\varphi_{\iota(i)} = (\varphi_i)^{-1}$.

In what follows, we will show that the **K**-theory of $\mathcal{flmod}_{\nu}\Lambda_{\nu}$ can be naturally identified with $\mathfrak{h}(\underline{\Delta} \setminus \nu\mathfrak{J})$, where the classes of simple objects $\{[S_i] \mid i \in \underline{\Delta} \setminus \nu\mathfrak{J}\}$ play the role of simple roots and the linear maps φ_i are precisely the ones induced by simple mutation functors Φ_i, Ψ_i.

§ 4.5 Intersection arrangements. An important tool in the study of root systems and Weyl group actions is the study of the dual representation. Let $\mathfrak{h}$ be the root lattice associated to $\underline{\Delta}$ as in the previous section, with the action of $W(\underline{\Delta})$ as described. The dual action on the Euclidean space $\Theta = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}, \mathbb{R})$ preserves each of the subsets $\{\delta > 0\}$, $\{\delta < 0\}$, and $\{\delta < 0\}$, and the fundamental domains for the respective actions have closures given by the cones

$$C^{+} = \{\theta \in \Theta \mid \theta(\alpha_i) \geqslant 0 \text{ for all } i \in \underline{\Delta}\}, \qquad C^{-} = -C^{+},$$
$$C^{0} = \{\theta \in \Theta \mid \theta(\alpha_i) \geqslant 0 \text{ for all } i \in \Delta, \quad \theta(\delta) = 0\}.$$

Further the faces of wC^{+} are all given by wC^{+}_{J} for some subset $J \subset \underline{\Delta}$, where

$$C^{+}_{J} = C^{+} \cap \bigcap_{i \in J} \{\alpha_i = 0\}.$$

The faces $wC^{-}_{J} \subseteq wC^{-}$ $(J \subseteq \underline{\Delta})$ and $wC^{0}_{J} \subseteq wC^{0}$ $(J \subseteq \Delta)$ are given likewise, and as a consequence we have a complete simplicial fan in Θ called the *Weyl arrangement* given as

$$\mathrm{Arr}(\underline{\Delta}) = \underbrace{\left\{wC^{+}_{J} \mid J \subseteq \underline{\Delta},\ w \in W(\underline{\Delta})\right\}}_{\mathrm{Arr}^{+}(\underline{\Delta})} \cup \underbrace{\left\{wC^{0}_{J} \mid J \subseteq \Delta,\ w \in W(\Delta)\right\}}_{\mathrm{Arr}(\Delta)} \cup \underbrace{\left\{wC^{-}_{J} \mid J \subseteq \underline{\Delta},\ w \in W(\underline{\Delta})\right\}}_{\mathrm{Arr}^{-}(\underline{\Delta})}. \tag{14}$$

The Weyl arrangement is induced by the root system for $\underline{\Delta}$, in the sense that every face can be described as the intersection of half-spaces defined by root hyperplanes $\{\alpha = 0\}$. Further each such hyperplane is corresponds to a unique positive root in $\mathrm{Root}^{+}(\underline{\Delta}) \cup \{\delta\}$.

The subfan $\mathrm{Arr}^{+}(\underline{\Delta})$ is supported on the rational subset $\{\delta > 0\} \cup \{0\}$, and is called the *Tits cone.* The subfan $\mathrm{Arr}(\Delta)$ is supported on $\{\delta = 0\}$, and induces a complete simplicial fan on this hyperplane. Further it has finitely many cones which are indexed over the parabolic subgroup $W(\Delta)$. This is possible because $W(\underline{\Delta})$ decomposes as a semi-direct product of $W(\Delta)$ and the coroot lattice, and the action on $\{\delta = 0\}$ is such that $W(\Delta)$ acts faithfully while the coroot lattice fixes the hyperplane pointwise.

We write $\mathrm{Cham}(\underline{\Delta})$ for the set of maximal cones (*chambers*) in $\mathrm{Arr}^{+}(\underline{\Delta})$, noting that subfans $\mathrm{Arr}^{+}(\underline{\Delta})$ and $\mathrm{Arr}^{-}(\underline{\Delta})$ are isomorphic so it suffices to consider just one of them. Likewise, we write $\mathrm{Cham}(\Delta)$ for the set of chambers in $\mathrm{Arr}(\Delta)$.

Now a subset $\mathfrak{I} \subset \underline{\Delta}$ with $|\underline{\Delta} \setminus \mathfrak{I}| \geqslant 2$ defines the restricted root lattice $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{I})$, which we view as a split quotient of $\mathfrak{h}$. Accordingly, the dual Euclidean space $\Theta(\underline{\Delta} \setminus \mathfrak{I}) = \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{I}), \mathbb{R})$ can be identified with the subspace $\bigcap_{i \in \mathfrak{I}} \{\alpha_i = 0\} \subset \Theta$. Since the subspace is defined by root hyperplanes, every cone $\sigma \in \mathrm{Arr}(\underline{\Delta})$ intersects $\Theta(\underline{\Delta} \setminus \mathfrak{I})$ in a face of σ, whence the Weyl arrangement induces a complete simplicial fan

$$\mathrm{Arr}(\underline{\Delta}, \mathfrak{I}) = \underbrace{\{\sigma \cap \Theta(\underline{\Delta} \setminus \mathfrak{I}) \mid \sigma \in \mathrm{Arr}^{+}(\underline{\Delta})\}}_{\mathrm{Arr}^{+}(\underline{\Delta}, \mathfrak{I})} \cup \underbrace{\{\sigma \cap \Theta(\underline{\Delta} \setminus \mathfrak{I}) \mid \sigma \in \mathrm{Arr}(\Delta)\}}_{\mathrm{Arr}(\Delta, \mathfrak{I})} \cup \underbrace{\{\sigma \cap \Theta(\underline{\Delta} \setminus \mathfrak{I}) \mid \sigma \in \mathrm{Arr}^{-}(\underline{\Delta})\}}_{\mathrm{Arr}^{-}(\underline{\Delta}, \mathfrak{I})}$$

called the (full) $\mathfrak{J}$-*cone arrangement.* This is precisely the subfan of $\mathrm{Arr}(\underline{\Delta})$ supported on $\Theta(\underline{\Delta} \setminus \mathfrak{J})$. It can be seen that the $\mathfrak{J}$-cone arrangement is induced by the restricted root system associated to $\mathfrak{J} \subset \underline{\Delta}$.

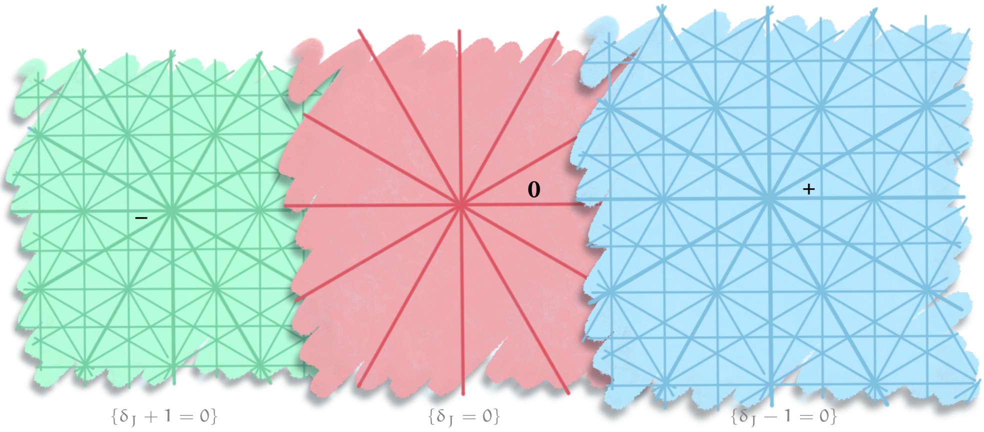


Figure 5. Representative affine slices of the $\mathcal{E}_{7,4}$ arrangement, associated to the Dynkin data ∘•×××•×× ($J = \{2, 3, 5, 6, 7\}$) as in fig. 4. The principal chambers C_J^+, C_J^0, C_J^- are indicated by the symbols **+**,**0**,**–** respectively.

The subfan $\mathrm{Arr}^+(\underline{\Delta}, \mathfrak{J})$, called the *Tits cone*, contains infinitely many cones and is supported on the subset $\{\delta_{\mathfrak{J}} > 0\} \cup \{0\}$. The subfan $\mathrm{Arr}(\Delta, \mathfrak{J})$ on the other hand is a complete simplicial fan on the hyperplane $\{\delta_{\mathfrak{J}} = 0\}$, with finitely many cones whose combinatorics are controlled by the ADE Dynkin diagram Δ. We write $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ for the subset of maximal cones in $\mathrm{Arr}^+(\underline{\Delta}, \mathfrak{J})$, and $\mathrm{Cham}(\Delta, \mathfrak{J})$ for the subset of maximal cones in $\mathrm{Arr}(\Delta, \mathfrak{J})$.

Simple wall crossings. Any chamber $wC^+ \in \mathrm{Cham}(\underline{\Delta})$ has $|\underline{\Delta}|$ codimension-one faces (*walls*), where the ith wall $wC^+_{\{i\}}$ spans the root hyperplane $\{w\alpha_i = 0\}$. The wall $wC^+_{\{i\}}$ is a face of precisely one other chamber, which lies on the 'other side' of this hyperplane. Examining (12) shows that this chamber is given by ws_iC^+, and we say ws_iC^+ is obtained from wC^+ by a *simple wall crossing.* Since every $w \in W(\underline{\Delta})$ is a product of simple reflections, it follows that any two chambers in $\mathrm{Cham}(\underline{\Delta})$ are connected by a finite sequence of simple wall crossings. The analogous statement holds for $\mathrm{Cham}(\Delta)$.

Iyama–Wemyss [IW, §1] show that while there may not be an underlying group action on the chambers of the $\mathfrak{J}$-cone arrangement, one can navigate between them by analogous wall-crossing combinatorics. To explain the construction, we first note that if $\sigma \in \mathrm{Arr}(\underline{\Delta})$ can be expressed as $wC_J^+ = w' C^+_{J'}$, then we must have $J = J'$ and $wW(J) = w'W(J')$ as cosets. Thus for each $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ there is a unique subset $\mathbb{J}(\sigma) \subset \underline{\Delta}$ such that σ can be expressed as $\sigma = wC^+_{\mathbb{J}(\sigma)}$. Further the codimension-one faces of σ are in bijection with $\underline{\Delta} \setminus \mathbb{J}(\sigma)$, where $i \in \underline{\Delta} \setminus \mathbb{J}(\sigma)$ determines the face $\sigma_i = wC_{\mathbb{J}(\sigma)+i}$. We then have the following.

Lemma 4.11 [IW, lemma 1.23]. *Given* $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *and* $i \in \underline{\Delta} \setminus \mathbb{J}(\sigma)$, *there is a unique cone* $\nu_i\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *which satisfies* $\nu_i\sigma \cap \sigma = \sigma_i$, *which we call the* simple wall crossing of σ at i. *Explicitly, if* $\sigma = wC_J^+$ *then this cone is given by*

$$\nu_i \cdot wC_J^+ = w \cdot w_J w_{J+i} \cdot C^+_{\nu_i J}.$$

This gives $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *the structure of a set with* $\underline{\Delta}$*-mutation, i.e. we have* $\mathbb{J}(\nu_i\sigma) = \nu_i\mathbb{J}(\sigma)$ *and* $\nu_{\iota(i)}\nu_i\sigma = \sigma$. *Moreover, this set has a connected exchange quiver (i.e. any two chambers in* $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *are connected by a finite sequence of simple wall crossings.)*

Analogously, $\mathrm{Cham}(\Delta, \mathfrak{J})$ *equipped with the data of simple wall crossings is a set with* Δ*-mutation, and this has a finite and connected exchange quiver.*

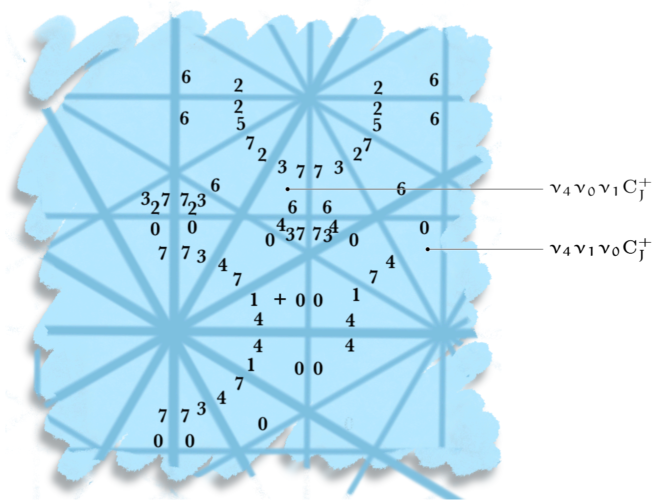
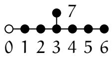


Figure 6. Continuing fig. 5, some wall crossings in the $\mathcal{E}_{7,4}$ Tits cone are indicated. Within each chamber σ, the walls are labelled by indices $i \in \underline{\Delta} \setminus \mathbb{J}(\sigma)$ such that crossing the i^{th} wall lands in the adjacent chamber $\nu_i\sigma$.

$\underline{\Delta} =$ 

The cone $C_{\mathfrak{J}}^+$ lies in the $\mathfrak{J}$-cone arrangement, and it can be seen that the simple wall crossing $\nu_i C_{\mathfrak{J}}^+$ is precisely the image of $C_{\nu_i\mathfrak{J}}^+ \in \mathrm{Cham}(\underline{\Delta}, \nu_i\mathfrak{J})$ under the map

$$(\varphi_i{}^{\vee})^{-1} : \Theta(\underline{\Delta} \setminus \nu_i\mathfrak{J}) \to \Theta(\underline{\Delta} \setminus \mathfrak{J})$$

transpose to the map (13). More generally, if $\nu = \nu_{i_n} \ldots \nu_{i_1}$ is an $\mathfrak{J}$-path then the cone $\nu C_{\mathfrak{J}}^+$ is the image of $C_{\nu\mathfrak{J}}^+ \in \mathrm{Cham}(\underline{\Delta}, \nu\mathfrak{J})$ under the map $((\varphi_{i_n} \circ \ldots \circ \varphi_{i_1})^{\vee})^{-1}$. Every chamber in the Tits cone of the $\mathfrak{J}$-cone arrangement has this form.

Simple wall crossing from a maximal cone in $\mathrm{Arr}^-(\underline{\Delta}, \mathfrak{J})$ is defined likewise. This is such that if ν is an $\mathfrak{J}$-path then we have $\nu C_{\mathfrak{J}}^- = -\nu C_{\mathfrak{J}}^+$, and every maximal cone in $\mathrm{Arr}^-(\underline{\Delta}, \mathfrak{J})$ is of this form.

Similarly each $\sigma \in \mathrm{Arr}(\underline{\Delta}, \mathfrak{J})$ can be assigned a unique subset $\mathbb{J}(\sigma) \subset \Delta$ such that we have $\sigma = wC^0_{\mathbb{J}(\sigma)}$. The codimension-one faces of σ are thus in bijection with $\Delta \setminus \mathbb{J}(\sigma)$, and a statement analogous to lemma 4.11 holds showing that $\mathrm{Cham}(\Delta, \mathfrak{J})$ is a set with Δ-mutation and a connected exchange quiver.

In particular if we write $C^0_{\mathfrak{J}} = C^0 \cap \Theta(\underline{\Delta} \setminus \mathfrak{J})$ (even when $\mathfrak{J}$ does not lie in Δ), we see that every $\sigma \in \mathrm{Cham}(\Delta, \mathfrak{J})$ can be written as $\nu C^0_{\mathfrak{J}}$ for some spherical $\mathbb{J}\left(C^0_{\mathfrak{J}}\right)$-path ν. We remark that $\mathbb{J}\left(C^0_{\mathfrak{J}}\right) = \mathfrak{J}$ if and only if $\mathfrak{J} \subset \Delta$.

It follows that if $\mathfrak{J} \subset \Delta$, then the $\mathfrak{J}$-cone arrangement can be explicitly described as

$$\mathrm{Arr}(\underline{\Delta}, \mathfrak{J}) = \underbrace{\bigcup_{\nu \text{ an } \mathfrak{J}\text{-path}} \mathrm{faces}\left(\nu C_{\mathfrak{J}}^+\right)}_{\mathrm{Arr}^+(\underline{\Delta}, \mathfrak{J})} \cup \underbrace{\bigcup_{\substack{\nu \text{ a spherical} \\ \mathfrak{J}\text{-path}}} \mathrm{faces}\left(\nu C_{\mathfrak{J}}^0\right)}_{\mathrm{Arr}^0(\underline{\Delta}, \mathfrak{J})} \cup \underbrace{\bigcup_{\nu \text{ an } \mathfrak{J}\text{-path}} \mathrm{faces}\left(\nu C_{\mathfrak{J}}^-\right)}_{\mathrm{Arr}^-(\underline{\Delta}, \mathfrak{J})}. \tag{15}$$

§ 4.6 The partial order on chambers. The $\mathfrak{J}$-cone arrangement is induced by the restricted root system, hence every chamber $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ is determined by the subset of positive roots which σ lies in the 'negative half-space' of, namely

$$[\sigma \leqslant 0] := \left\{\alpha \in \mathrm{Root}^+(\underline{\Delta}, \mathfrak{J}) \mid \sigma \subset \{\alpha \leqslant 0\}\right\}.$$

This assignment gives a partial order $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ given by

$$\sigma \leqslant \sigma' \quad \iff \quad [\sigma \leqslant 0] \subseteq [\sigma' \leqslant 0].$$

Clearly the chamber $C_{\mathfrak{J}}^+$ is minimal with respect to this order since we have $[C_{\mathfrak{J}}^+ \leqslant 0] = \emptyset$, and hence for a general $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$, the set $[\sigma \leqslant 0]$ determines the hyperplanes which 'separate' σ and $C_{\mathfrak{J}}^+$. We now exhibit various properties of this partial order.

Lemma 4.12. *Bounded intervals in the poset* $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *are finite, i.e. if* $\sigma, \sigma' \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *satisfy* $\sigma' \leqslant \sigma$, *then there are finitely many chambers in the interval* $[\sigma', \sigma]$.

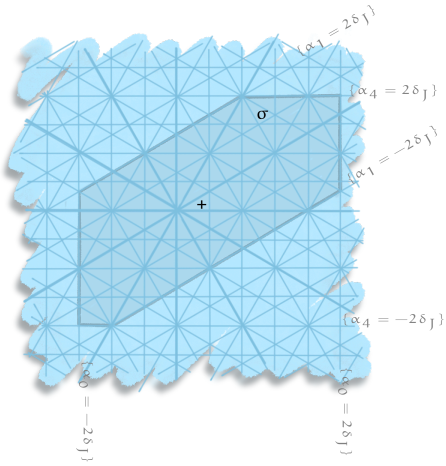
Proof. It suffices to show the set $[\sigma \leqslant 0]$ is finite for any $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$, i.e. σ and $C^+_{\mathfrak{J}}$ are separated by finitely many hyperplanes. Now we can write the Tits cone as the nested union of 'boxes'

$$\{\delta_{\mathfrak{J}} > 0\} \cup \{0\} = \bigcup_{N \geqslant 0} \{\theta \in \Theta(\underline{\Delta} \setminus \mathfrak{J}) \mid -N\delta_{\mathfrak{J}}(\theta) \leqslant \alpha_i(\theta) \leqslant N\delta_{\mathfrak{J}}(\theta) \quad \text{for all } i \in \underline{\Delta} \setminus \mathfrak{J}\}.$$

In particular, both σ and $C^+_{\mathfrak{J}}$ lie in some sufficiently large box

$$\bigcap_{i \in \underline{\Delta} \setminus \mathfrak{J}} \{\alpha_i + N\delta_{\mathfrak{J}} \geqslant 0\} \cap \{\alpha_i - N\delta_{\mathfrak{J}} \leqslant 0\}$$

for $N \gg 0$ (see figure beside for a representative box in the $\mathcal{E}_{7,4}$ Tits cone with $N = 2$). But each box is cut only by finitely many root hyperplanes, since the intersection of any box with the level set $\{\delta_{\mathfrak{J}} = 1\}$ is compact and the root hyperplanes form a locally finite arrangement away from the origin. The conclusion then follows since any hyperplane separating σ and $C^+_{\mathfrak{J}}$ must pass through this box. □



The boxes appearing in the above proof also provide upper bounds on the sizes of intervals in the partial order. Building upon this notion, we make the following definition.

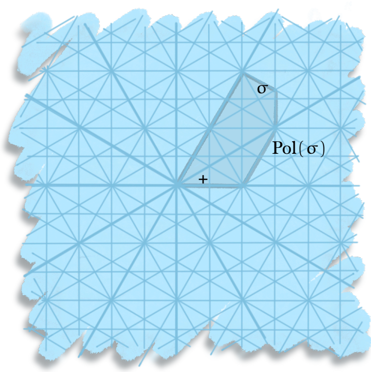
Definition 4.13. The *lower polytope* of $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$, written $\mathrm{Pol}(\sigma)$, is the subset of $\Theta(\underline{\Delta}, \mathfrak{J})$ given by the union of cones $\bigcup_{\sigma' \leqslant \sigma} \sigma'$.

Lemma 4.14. *The lower polytope of any chamber* $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *is a convex polyhedral cone.*

Proof. Write $[\sigma \geqslant 0] = \mathrm{Root}^+(\underline{\Delta}, \mathfrak{J}) \setminus [\sigma \leqslant 0]$ for the collection of positive roots whose associated hyperplane does not separate σ and $C^+_{\mathfrak{J}}$. Then a chamber σ' satisfies $\sigma' \leqslant \sigma$ if and only if we have $[\sigma' \geqslant 0] \supseteq [\sigma \geqslant 0]$, i.e. σ' lies in the half space $\{\alpha \geqslant 0\}$ for every $\alpha \in [\sigma \geqslant 0]$. Since any intersection of root half-spaces is a union of chambers, it follows that we have

$$\mathrm{Pol}(\sigma) = \bigcap_{\alpha \in [\sigma \geqslant 0]} \{\alpha \geqslant 0\},$$

and hence the lower polytope being an intersection of convex sets is convex. Since it is the union of finitely many polyhedral cones, it is one too. □



The poset $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ is also particularly amiable to the study of paths and covering relations.

Lemma 4.15. *For each* $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *and* $i \in \underline{\Delta} \setminus \mathfrak{J}(\sigma)$*, we either have* $\sigma \lessdot \nu_i\sigma$ *or* $\sigma \gtrdot \nu_i\sigma$.

Proof. By definition of simple wall crossing there is precisely one hyperplane that separates σ and $\nu_i\sigma$, i.e. σ and $\nu_i\sigma$ lie on the same side of all root hyperplanes except for one. Thus one of the sets $[\sigma \leqslant 0], [\nu_i\sigma \leqslant 0]$ is contained in the other, and their difference contains exactly one element. The result follows. □

Lemma 4.16. *Given* $\sigma \in \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *and a positive path* $\nu = \nu_{i_n} \ldots \nu_{i_1}$ *from* σ *such that* $\sigma \leqslant \nu\sigma$*, the following are equivalent.*

(1) ν *is* minimal*, i.e.* ν *has minimal length among all positive paths from* σ *to* $\nu\sigma$.

(2) ν *is* reduced, *i.e.* ν *crosses each root hyperplane at most once.*

(3) the length of ν *is equal to the number of hyperplanes separating* σ *and* $\nu\sigma$*, i.e. the set* $[\nu\sigma \leqslant 0] \setminus [\sigma \leqslant 0]$ *contains* n *elements.*

(4) ν *is* atomic*, i.e. successive truncations of* ν *give us a sequence* $\sigma < \nu_{i_1}\sigma < \nu_{i_2}\nu_{i_1}\sigma < \ldots < \nu\sigma$.

Proof. Since the arrangement of root hyperplanes is locally finite away from 0, we have the equivalence (1) $\iff$ (2) [see for example Sal87, lemma 2]. Further, the implications (4)$\Rightarrow$(3)$\Rightarrow$(1) are immediate where for the latter we note that any positive path must cross each separating hyperplane at least once. Lastly, suppose ν is reduced but not atomic, i.e. we have $\nu_{i_{j-1}}\ldots\nu_{i_1}\sigma > \nu_{i_j}\ldots\nu_{i_1}\sigma$ for some j. Thus there is a positive root α such that ν passes from $\{\alpha \leqslant 0\}$ into $\{\alpha \geqslant 0\}$, and since this hyperplane is crossed exactly once, we therefore have $\alpha \in [\sigma \leqslant 0]$ but $\alpha \notin [\nu\sigma \leqslant 0]$. This contradicts $\sigma \leqslant \nu\sigma$, thus showing (2)$\Rightarrow$(4). ☐

Now any two chambers in a locally finite hyperplane arrangement can be connected by a reduced path. It follows that every relation in the poset $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ is realised by the Hasse quiver, i.e. we have $\sigma < \sigma'$ if and only if there is a chain of covering relations $\sigma = \sigma^0 \lessdot \ldots \lessdot \sigma^n = \sigma'$. This also shows that every covering relation in the poset must arise from simple wall crossing as in lemma 4.15, so that the Hasse quiver is a subquiver of the exchange quiver in the obvious way and a positive path is atomic if and only if it lies in the Hasse quiver.

Since the Hasse quiver of $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ realises all relations, we have the following consequence.

Proposition 4.17. *Suppose* $(P, \leqslant)$ *is a poset and* $f : \mathrm{Cham}(\underline{\Delta}, \mathfrak{J}) \to P$ *is a bijection that induces an isomorphism of Hasse quivers. Then* f *is an isomorphism of posets.*

§ 4.7 Mutation functors and Grothendieck groups. We now show that the **K**-theoretic maps induced by the mutation functors Φ_ν, Ψ_ν obey the same combinatorial rules which govern mutations of root lattices. To explain this, note that for any $\mathfrak{J}$-path ν, theorem 3.9 gives a unique way to index the simples of $\mathit{flmod}\,{}_\nu\Lambda_\nu$ over the vertices of $\underline{\Delta} \setminus \nu\mathfrak{J}$. Further, $\mathfrak{J}$ is a proper subset of Δ by construction, and hence we see that $|\underline{\Delta} \setminus \nu\mathfrak{J}| = |\underline{\Delta} \setminus \mathfrak{J}| \geqslant 2$. Thus the restricted root lattice $\mathfrak{h}(\underline{\Delta} \setminus \nu\mathfrak{J})$ is defined and can be identified with $\mathbf{K}\,{}_\nu\Lambda_\nu$ by mapping the classes of simples to simple restricted roots. We then have the following.

Lemma 4.18. *For* $i \in \underline{\Delta} \setminus \mathfrak{J}$*, the simple mutation functors* $\Phi_i, \Psi_i : \mathbf{D}^{\mathrm{fl}}\,{}_{\nu_i}\Lambda_{\nu_i} \to \mathbf{D}^{\mathrm{fl}}\Lambda$ *both induce the same map on* **K***-theory and this agrees with the map* $\varphi_i : \mathfrak{h}(\underline{\Delta} \setminus \nu_i\mathfrak{J}) \to \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$ *as in* (13).

The proof, which fleshes out [NW23, remark 5.2], uses silting theory of contracted preprojective algebras, and is deferred until the end of this subsection. Combined with corollary 4.9, the above lemma immediately shows that if $\nu = \nu_{i_n}\ldots\nu_{i_1}$ is a $\mathfrak{J}$-path then the map on **K**-theory induced by Φ_ν and Ψ_ν agrees with the composite $\varphi_{i_n} \circ \ldots \circ \varphi_{i_1}$. In particular, we have the following.

Corollary 4.19. *If* ν *is a* $\mathfrak{J}$*-path and* S_j *is a simple* ${}_\nu\Lambda_\nu$*-module, then the object* $\Phi_\nu(S_j)$ *(resp.* $\Psi_\nu(S_j)$*) has* **K***-theory class given by a primitive positive restricted root in* $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$.

More importantly, we can use lemma 2.10 and the discussion following lemma 4.11 to read off the dual cones of hearts in $\mathrm{tilt}^{\pm}(H)$ which are obtained from $H = \mathit{flmod}\,\Lambda$ by mutation.

Theorem 4.20. *For each* $\mathfrak{J}$*-path* ν*, the cones* $\nu C^{\pm}_{\mathfrak{J}} \subset \Theta(\underline{\Delta} \setminus \mathfrak{J})$ *are heart cones in* $\mathrm{HFan}(H)$*. In particular, these are described as*

$$\mathbf{C}(\Psi_\nu H) = \nu C^+_{\mathfrak{J}}, \qquad \mathbf{C}(\Phi_\nu H[-1]) = \nu C^-_{\mathfrak{J}}$$

and therefore the associated numerical torsion theories and intermediate hearts are

$$\begin{aligned} &H_{\mathrm{tr}}(\nu C^+_{\mathfrak{J}}) = H^{\mathrm{tr}}(\nu C^+_{\mathfrak{J}}) = U_\nu, && H_{\mathrm{tf}}(\nu C^+_{\mathfrak{J}}) = H^{\mathrm{tf}}(\nu C^+_{\mathfrak{J}}) = V_\nu, && H_{\mathrm{tt}}(\nu C^+_{\mathfrak{J}}) = H^{\mathrm{tt}}(\nu C^+_{\mathfrak{J}}) = \Psi_\nu H, \\ &H_{\mathrm{tr}}(\nu C^-_{\mathfrak{J}}) = H^{\mathrm{tr}}(\nu C^-_{\mathfrak{J}}) = T_\nu, && H_{\mathrm{tf}}(\nu C^-_{\mathfrak{J}}) = H^{\mathrm{tf}}(\nu C^-_{\mathfrak{J}}) = F_\nu, && H_{\mathrm{tt}}(\nu C^-_{\mathfrak{J}}) = H^{\mathrm{tt}}(\nu C^-_{\mathfrak{J}}) = \Phi_\nu H[-1]. \end{aligned}$$

Corollary 4.21 (Theorem 4.1 (1)). *Every algebraic heart in* $\mathrm{tilt}(H)$ *lies in exactly one of the subsets* $\mathrm{tilt}^+(H)$ *or* $\mathrm{tilt}^-(H)$, *i.e. is either equal to* $\Psi_v H$ *or to* $\Phi_v H[-1]$ *for some* $\mathfrak{J}$*-path* v. *Thus we have disjoint-union decompositions*

$$\text{alg-tilt}(H) = \mathrm{tilt}^+(H) \sqcup \mathrm{tilt}^-(H), \qquad \mathrm{ftors}(H) = \mathrm{tors}^+(H) \sqcup \mathrm{tors}^-(H), \qquad \mathrm{ftorf}(H) = \mathrm{torf}^+(H) \sqcup \mathrm{torf}^-(H).$$

Proof. If $K \in \mathrm{tilt}(H)$ is algebraic, then by theorem 2.11 we have that $\mathbf{C}(K)$ is a full dimensional simplicial cone in $\mathrm{HFan}(H)$ and further K the unique intermediate heart with this heart cone. But from theorem 4.20 we see that the union of Tits cones

$$\bigcup \mathrm{faces}(vC^+_{\mathfrak{J}}) \quad \cup \quad \bigcup \mathrm{faces}(vC^-_{\mathfrak{J}})$$

defines a subfan of $\mathrm{HFan}(H)$ supported on the complement of the hyperplane $\Theta^0(\underline{\Delta} \setminus \mathfrak{J})$. The cone $\mathbf{C}(K)$, being full-dimensional, lies in this subfan and therefore must be of the form $vC^{\pm}_{\mathfrak{J}}$, giving the result. □

We conclude with a proof of lemma 4.18. The connection to combinatorics comes by reducing to surfaces as in [Wem18, §5.3], so we consider a general elephant $\mathrm{Spec}\,\overline{R} \hookrightarrow \mathrm{Spec}\,R$ and the reduction functor $F = (-) \otimes_R \overline{R}$. It is then known that $F\Lambda$ is a contracted preprojective algebra with indecomposable projective modules naturally indexed over $\underline{\Delta} \setminus \mathfrak{J}$, thus giving an identification of $\mathbf{K}_{\mathrm{split}}(\mathit{proj}\, F\Lambda) \otimes \mathbb{R}$ with $\Theta(\underline{\Delta} \setminus \mathfrak{J})$ where the dual basis to simple roots gives classes of indecomposable projectives. Here Iyama–Wemyss [IW] show that the silting fan of $F\Lambda$ coincides with the fan $\mathrm{Arr}^+(\underline{\Delta}, \mathfrak{J}) \cup \mathrm{Arr}^-(\underline{\Delta}, \mathfrak{J})$.

By [Gar23, proposition B] the reduction F is compatible with silting mutation and induces an isomorphism of silting fans via the induced isomorphism $\mathbf{K}_{\mathrm{split}}(\mathit{proj}\,\Lambda) \cong \mathbf{K}_{\mathrm{split}}(\mathit{proj}\,F\Lambda)$. Further this identification of $\mathbf{K}_{\mathrm{split}}(\mathit{proj}\,\Lambda) \otimes \mathbb{R}$ with $\Theta(\underline{\Delta} \setminus \mathfrak{J})$ is compatible with that of $\mathbf{K}\,\Lambda$ with $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$, so the silting fan of Λ is naturally identified with the fan $\mathrm{Arr}^+(\underline{\Delta}, \mathfrak{J}) \cup \mathrm{Arr}^-(\underline{\Delta}, \mathfrak{J})$. We then have the following.

Proof of lemma 4.18. By lemmas 4.3 and 4.5 we have $[\Psi_i S_j] = [\Phi_i S_j]$ for all $j \in \underline{\Delta} \setminus \mathfrak{J}$, so both Φ_i and Ψ_i induce the same map on $\mathbf{K}$-theory which we momentarily call $\psi_i : \mathfrak{h}(\underline{\Delta} \setminus v_i\mathfrak{J}) \to \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$.

Now the functor Φ_i comes from a derived equivalence between Λ and ${}_{v_i}\Lambda_{v_i}$, and the induced map

$$\mathbf{K}_{\mathrm{split}}(\mathit{proj}\,{}_{v_i}\Lambda_{v_i}) \otimes \mathbb{R} \longrightarrow \mathbf{K}_{\mathrm{split}}(\mathit{proj}\,\Lambda) \otimes \mathbb{R}$$

is precisely $(\psi_i^\vee)^{-1}$. Further this map respects silting theory and gives a map of silting fans. In particular, the image of $C^+_{v_i\mathfrak{J}}$ under $(\psi_i^\vee)^{-1}$ is a cone in the intersection arrangement of $\Theta(\underline{\Delta} \setminus \mathfrak{J})$.

By lemma 4.3 we have $[\Phi_i S_{\iota(i)}] = -[S_i]$ and hence $\psi_i(\alpha_{\iota(i)}) = -\alpha_i$. Likewise if $j \neq \iota(i)$ then lemma 4.5 shows $\psi_i \alpha_j = b_{ij}\alpha_i + \alpha_j$ for some non-negative integer b_{ij} recording the multiplicity of N_j in a minimal right $\mathrm{add}(N/N_i)$-approximation of N_i.

Thus image of $C^+_{v_i\mathfrak{J}}$ in $(\psi_i^\vee)^{-1}$ intersects (and hence is equal to) the cone $v_i C^+_{\mathfrak{J}}$, and the images of primitive integral generators are primitive integral. Evidently from (13) $(\varphi_i^\vee)^{-1}$ admits the same description, hence it follows that $\psi_i = \varphi_i$ as required. □

§ 4.8 The poset of intermediate algebraic hearts. Continuing our analysis of torsion theories on $H = \mathit{flmod}\,\Lambda$, theorem 4.20 shows that simply identifying $\mathfrak{J}$-paths gives us bijections between $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ and the posets $\mathrm{tilt}^+(H)$, $\mathrm{tilt}^-(H)$, and therefore also $\mathrm{MM}^N(R)$. We now show that these bijections induce isomorphisms of Hasse quivers, whence proposition 4.17 shows the bijections give isomorphisms of posets.

Write $\mathbf{C}$ for the composite bijection $\mathrm{MM}^N(R) \to \mathrm{tilt}^+(H) \to \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$, i.e. the assignment $\mathbf{C}(vN) = vC^+_{\mathfrak{J}}$.

Lemma 4.22. *The map* $\mathbf{C} : \mathrm{MM}^N(R) \to \mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$ *is an anti-isomorphism of posets, i.e. we have* $M \leqslant M'$ *in* $\mathrm{MM}^N(R)$ *if and only* $\mathbf{C}(M) \geqslant \mathbf{C}(M')$ *in* $\mathrm{Cham}(\underline{\Delta}, \mathfrak{J})$.

Proof. Following proposition 4.17, it suffices to check the behaviour of the map $\mathbf{C}$ on covering relations. Thus suppose $M < M'$ is a covering relation in $\mathrm{MM}^N(R)$, in particular we must have $M' = v_i M$ for some $i \in \underline{\Delta} \setminus \mathbb{J}(M)$. Writing $M = vN$ for some $\mathfrak{J}$-path v, we need to show we have $vC^+_{\mathfrak{J}} > v_i v C^+_{\mathfrak{J}}$ i.e. $vC^+_{\mathfrak{J}}$ and $C^+_{\mathfrak{J}}$ lie on the same side of the hyperplane spanned by $vC^+_{\mathfrak{J}} \cap v_i v C^+_{\mathfrak{J}}$.

Now by theorem 4.6 the object $\Psi_\nu S_i$ lies in the hearts H, $\Psi_\nu H$, and $\Psi_{\nu_i\nu}H[1]$. Thus the heart cones $\mathbf{C}(\Psi_\nu H) = \nu C_{\mathfrak{J}}^+$ and $\mathbf{C}(\Psi_{\nu_i\nu}H) = \nu_i\nu C_{\mathfrak{J}}^+$ lie on the opposite sides of the hyperplane which is orthogonal to the **K**-theory class $\beta_i = [\Psi_\nu S_i]$, and further σ lies on the same side of the hyperplane as $\mathbf{C}(H) = C_{\mathfrak{J}}^+$. Lastly, since $\nu C_{\mathfrak{J}}^+$ and $\nu_i\nu C_{\mathfrak{J}}^+$ intersect in a cone of codimension 1, the span of this wall must necessarily be the hyperplane $\{\beta_i = 0\}$. □

The following is now immediate.

Theorem 4.23. *There are natural isomorphisms between the posets* $(\mathrm{MM}^N(R), \leqslant)$, $(\mathrm{tilt}^+(H), \leqslant)$, $(\mathrm{tilt}^-(H), \leqslant)^{\mathrm{op}}$, *and* $(\mathrm{Cham}(\underline{\Delta}, \mathfrak{J}), \leqslant)^{\mathrm{op}}$, *given by identifying positive paths from the unique maximal element.*

In particular, we deduce the important conclusion that all bounded intervals in $\mathrm{MM}^N(R)$ (and therefore all bounded intervals in $\mathrm{tors}^\pm(H)$, $\mathrm{torf}^\pm(H)$, $\mathrm{tilt}^\pm(H)$) are finite. This observation gives us the following.

Corollary 4.24 (Theorem 4.1 (2) and (3)). *If* $U \in \mathrm{tors}(H)$ *is a torsion class satisfying* $U_\nu \subseteq U \subseteq U_{\nu'}$ *for* $\mathfrak{J}$*-paths* ν *and* ν'*, then we must have* $U = U_{\nu''}$ *for some* $\mathfrak{J}$*-path* ν''*. Thus given torsion classes* $U_\nu \subseteq U_{\nu'}$*, the interval of torsion theories* $[U_\nu, U_{\nu'}]$ *lies entirely in* $\mathrm{tors}^+(H)$ *and is therefore finite.*

The analogous statement holds for $\mathrm{tors}^-(H)$.

Proof. Given $U_\nu \subseteq U \subseteq U_{\nu'}$, consider the collection of torsion classes

$$\left\{U_\mu \in \mathrm{tors}^+(H) \mid U_\mu \subseteq U\right\}.$$

This is non-empty (since $U_\nu \subseteq U$) and finite (since $U_\mu \subseteq U$ implies μN lies in the finite interval $[\nu' N, N]$ in $\mathrm{MM}^N(R)$), so the set has some maximal element $U_{\nu''}$. We claim that $U = U_{\nu''}$ in this case. Indeed if not, then $U \cap V_{\nu''}$ is a non-zero torsion class in $\Psi_{\nu''}(H)$ whence it contains at least one simple object $\Psi_{\nu''}(S_i) \in \Psi_{\nu''}(H)$. But this contradicts the maximality of $U_{\nu''}$, since we have

$$U_{\nu''} \quad \subseteq \quad U_{\nu''} * \langle \Psi_{\nu''}(S_i)\rangle \;=\; U_{\nu_i\nu''} \quad \subseteq \quad U.$$

To obtain the analogous statement for $\mathrm{tors}^-(H)$, we repeat the argument with the torsion-free classes F_ν. □

The saturation of alg-tilt(H) also allows us to rule out the possibility of non-algebraic hearts having non-zero heart cones in $\mathrm{Arr}^+(\underline{\Delta}, \mathfrak{J}) \cup \mathrm{Arr}^-(\underline{\Delta}, \mathfrak{J})$.

Corollary 4.25. *Let* $\sigma \in \mathrm{Arr}(\underline{\Delta}, \mathfrak{J})$ *be a cone not contained in the hyperplane* $\{\delta_{\mathfrak{J}} = 0\}$*. Then* σ *is the heart cone of some* $K \in \mathrm{tilt}(H)$ *if and only if* σ *is full-dimensional and* K *is an algebraic tilt of* H.

Proof. One implication is clear from theorem 4.20, so suppose σ in $\mathrm{Arr}(\underline{\Delta}, \mathfrak{J}) \setminus \mathrm{Arr}(\Delta, \mathfrak{J})$ is the heart cone of some heart in tilt(H). We may assume σ lies in $\mathrm{Arr}^+(\underline{\Delta}, \mathfrak{J})$.

By corollary 4.24 it suffices to show $H_{tt}\sigma$ and $H^{tt}\sigma$ both lie in $\mathrm{tilt}^+(H)$, so we consider the set

$$[H_{tt}\sigma, H^{tt}\sigma] \cap \text{alg-tilt}(H) = \{K \in \text{alg-tilt}(H) \mid \sigma \subset \mathbf{C}\,K\}.$$

Since the $\mathfrak{J}$-cone arrangement is locally finite away from the origin, this is a finite subset of $\mathrm{tilt}^+(H)$. In particular it has maximal and minimal elements K^σ, K_σ respectively.

If $H_{tt}\sigma$ is not algebraic, then the inequality $K_\sigma > H_{tt}\sigma$ is strict and by corollary 4.7, this relation is not covering. Thus there is some heart $K' \in \mathrm{tilt}^+(H)$ satisfying $K_\sigma > K' > H_{tt}\sigma$. But then K' evidently lies in $[H_{tt}\sigma, H^{tt}\sigma]$, this contradicts the minimality of K_σ. An analogous argument shows $H^{tt}\sigma$ is algebraic as required. □

This completes our discussion of the poset of functorially finite torsion theories in H.

§ 5 Limits of Silting theory

Whilst the algebraic context emerging from Van den Bergh's equivalence (4) allows us to completely classify algebraic t-structures intermediate with respect to $\mathscr{per}(\frac{X}{Z})$ (theorem 4.1), the existence of non-algebraic intermediate hearts is evident– for instance by observing that the heart cones of algebraic hearts do not fill out the vector space $\Theta(\Delta \setminus \mathfrak{J})$, but perhaps more obviously by noting that $\mathscr{coh}\,X = \mathscr{Coh}\,X \cap \mathbf{D}^0 X$ is a non-Artinian category that is the heart of a t-structure intermediate with respect to $\mathscr{per}(\frac{X}{Z})$.

This section studies the (necessarily non-algebraic) t-structures whose heart cone lies outside the Tits cones $\mathrm{Arr}^+(\Delta, \mathfrak{J}) \cup \mathrm{Arr}^-(\Delta, \mathfrak{J})$, thus obtaining a complete description of the tilts of $\mathscr{per}(\frac{X}{Z})$ which have non-zero heart cones. In particular we show that every maximal cone in $\mathrm{Arr}(\Delta, \mathfrak{J})$ is the heart cone of an intermediate t-structure of geometric origin, thereby also showing that the heart fan $\mathrm{HFan}(\mathscr{per}(\frac{X}{Z}))$ must coincide with the $\mathfrak{J}$-cone arrangement $\mathrm{Arr}(\Delta, \mathfrak{J})$ (corollary 5.18).

Flops. The geometric t-structures naturally arise from birational modifications of X, where surgery operations described below replace exceptional curves in a way that preserves the derived category.

Theorem 5.1 [Che02, theorem 1.1]. *Suppose Y is a normal 3-fold with at worst terminal Gorenstein singularities, and the maps $\tau : X \to Y$, $\tau' : W \to Y$ are flopping contractions over Y such that whenever D is a τ-nef divisor on X, the proper transform of $-D$ across the birational map $X \dashrightarrow W$ is $\mathbb{Q}$-cartier and τ'-nef. Then there is a Fourier–Mukai type equivalence of derived categories $\mathbf{D}^b W \to \mathbf{D}^b X$.*

A map $\tau' : W \to Y$ satisfying the given properties, if it exists, is unique and is called the *flop* of $\tau : X \to Y$. We say the equivalence $\mathbf{D}^b W \to \mathbf{D}^b X$ above is the *Bridgeland–Chen flop functor.*

Remark 5.2. There are various equivalent definitions of a flop in the literature, see for example [Kol90]. Given a flopping contraction $\tau : X \to Y$ between normal 3-folds, one typically needs to choose a $\mathbb{Q}$-cartier $\mathbb{Q}$-divisor D on X such that $-D$ is τ-ample in order to define the D-flop (i.e. the $(K_X + D)$-flip) of τ, which is unique and independent of the choice of D by [KM98, corollary 6.4]. In the special case that X (equivalently Y) is terminal, the existence of the flop is proved in theorem 6.14 *ibid.*

Now the flopping contraction $\pi : X \to Z$ of interest to us is over a complete local base, and therefore [Sch01, §2] guarantees that we can freely contract exceptional curves. That is to say, given any collection of exceptional curves $C_I = \bigcup_{i \in I} C_i \subset X$, the map π admits a unique factorisation

$$X \xrightarrow{\ \tau\ } Y \xrightarrow{\ \varpi\ } Z, \qquad \pi = \varpi \circ \tau \tag{16}$$

where τ contracts C_i to a point if and only if $i \in I$. We say τ is a *partial contraction over* Z. Since the singularity on Z is assumed to be isolated, it follows that both τ and ϖ are flopping contractions.

In particular we can consider the flop $\tau' : W \to Y$ of τ, and track the category $\mathscr{coh}\,W$ across the Bridgeland–Chen flop functor to get a new t-structure on $\mathbf{D}^0 X$. Further, the map $\varpi \circ \tau' : W \to Z$ is itself a flopping contraction over Z so iteratively flopping exceptional curves gives new hearts on $\mathbf{D}^0 X$.

The limiting hyperplane. In § 5.3 we show that each geometric heart constructed using iterated flops is intermediate with respect to $H = \mathscr{per}(\frac{X}{Z})$, and is the supremum (in $\mathrm{tilt}(H)$) of an appropriate collection of algebraic hearts. This uses the interplay of flops and mutation-combinatorics (§§ 5.1 and 5.2) to naturally identify the hyperplane $\{\delta_{\mathfrak{J}} = 0\} \subset \Theta(\Delta \setminus \mathfrak{J})$ with the cone of movable divisors on X. Thus the silting theory of Λ is 'completed' by the birational geometry of X.

Now Van den Bergh's construction of a tilting bundle $\mathscr{V}(\frac{X}{Y})$ on X [Van04] works in the generality of crepant partial contractions as in (16), allowing Y to be enhanced with a sheaf of non-commutative algebras $\mathscr{A} = \tau_* \mathscr{End}\,\mathscr{V}(\frac{X}{Y})$ such that the 'non-commutative scheme' $(Y, \mathscr{A})$ is derived equivalent to $(X, \mathscr{O}_X)$. Thus $\mathbf{D}^0 X$ has a heart $\mathscr{per}(\frac{X}{Y})$ that is equivalent to the category of coherent $\mathscr{A}$-modules.

In § 5.4 we study the structure of $\mathcal{per}(\frac{X}{Y})$ and modifications thereof, using the combinatorics of flops and mutations to enumerate them and characterise when these are intermediate with respect to $\mathcal{per}(\frac{X}{Z})$. In particular, we show in § 5.6 that $\mathcal{per}(\frac{X}{Y})$ sits in $\operatorname{tilt}(H)$ as the supremum of all geometric hearts $\mathcal{coh}\, W$ corresponding to birational models W that admit a partial contraction to Y, i.e. are obtained by iteratively flopping curves that get contracted by the map $\tau : X \to Y$.

The heart cone $\mathbf{C}(\mathcal{per}(\frac{X}{Y}))$ thus naturally is the intersection of the corresponding geometric heart cones. This realises non-zero cones in $\operatorname{Arr}(\Delta, \mathfrak{J})$ as precisely the heart cone of (semi-)geometric intermediate hearts (corollary 5.29), thus concluding the analysis of all intermediate hearts with non-zero heart cones.

Combinatorics of flops (§§ 5.1 to 5.3)

§ 5.1 Flops and mutation. Wemyss' homological minimal model programme [Wem18] addresses the question of classifying *birational models* of X, i.e. varieties W obtained by iteratively flopping subsets of π-exceptional curves. Abusing notation to write $\pi : W \to Z$ for the corresponding flopping contraction, we can again consider the associated vector bundle $\mathcal{V}(\frac{W}{Z})$, the modifying R-module generator $M = \pi_* \mathcal{V}(\frac{W}{Z})$, and the equivalence

$$\mathrm{VdB} : \mathbf{D}^{\mathrm{b}}(\operatorname{End} M) \longrightarrow \mathbf{D}^{\mathrm{b}} W$$

constructed as in § 3. In this context, Wemyss shows that flops obey mutation-combinatorics following the modifying generators.

Theorem 5.3 [Wem18]. *If $\pi : W \to Z$ is a birational model of $\pi : X \to Z$, then the associated modifying R-module generators $M = \pi_* \mathcal{V}(\frac{W}{Z})$ and $N = \pi_* \mathcal{V}(\frac{X}{Z})$ lie in the same mutation class, and every module in $\operatorname{MMG}^N(R)$ arises in this way from a unique birational model of X.*

Thus the set $\operatorname{Bir}(\frac{X}{Z})$ of birational models of X is in bijection with $\operatorname{MMG}^N(R)$ via the map $W \mapsto \pi_* \mathcal{V}(\frac{W}{Z})$. Since $\operatorname{MMG}^N(R)$ is a set with Δ-mutation, $\operatorname{Bir}(\frac{X}{Z})$ inherits this structure via the above bijection so in particular for any $W \in \operatorname{Bir}(\frac{X}{Z})$ with associated modifying generator $\pi_* \mathcal{V}(\frac{W}{Z}) = M$ we have $\mathbb{J}(W) = \mathbb{J}(M)$, and for any $\mathbb{J}(W)$-path ν the birational model νW is the unique element of $\operatorname{Bir}(\frac{X}{Z})$ satisfying $\pi_* \mathcal{V}(\frac{\nu W}{Z}) = \nu M$.

This mutation structure on $\operatorname{Bir}(\frac{X}{Z})$ can be realised intrinsically as arising from flops– by theorem 3.9 the indecomposable summands of M are indexed over $\underline{\Delta} \setminus \mathbb{J}(M)$ in a way that applying ν_i corresponds to mutating the ith summand, and the only free summand is $M_0 \cong R$. It follows that the non-free indecomposable summands of $\mathcal{V}(\frac{W}{Z})$ are indexed over $\Delta \setminus \mathbb{J}(W)$. By lemma 3.8 these summands are naturally in bijection with the integral exceptional curves of W and exceptional curves is bijective.

Therefore we see that the reduced exceptional fiber of any birational model $\pi : W \to Z$ can be written as a union of integral curves $\bigcup_{i \in \Delta \setminus \mathbb{J}(W)} C_i$, and the following result of Wemyss realises the operation ν_i on $\operatorname{Bir}(\frac{X}{Z})$ as a flop of the ith curve.

Theorem 5.4 [Wem18, theorem 4.2]. *Fix $i \in \Delta \setminus \mathfrak{J}$, and suppose $\nu_i X = W \to Z$ is the birational model of X corresponding to the modifying R-module generator $M = \nu_i N$ with reduced exceptional fiber $\bigcup_{i \in \Delta \setminus \nu_i \mathfrak{J}} C_i$ indexed as above. Then W is precisely the flop of X in the curve C_i, and $C_{\iota(i)} \subset W$ is the proper transform of $C_i \subset X$. Further, in this case the Bridgeland–Chen flop functor $\mathbf{D}^{\mathrm{b}} W \to \mathbf{D}^{\mathrm{b}} X$ coincides with the composite equivalence*

$$\mathbf{D}^{\mathrm{b}} W \xrightarrow{\mathrm{VdB}^{-1}} \mathbf{D}^{\mathrm{b}}(\operatorname{End} M) \xrightarrow{\Psi_i(-) = \mathbf{R}\mathrm{Hom}(\mathrm{Hom}_R(M,N),-)} \mathbf{D}^{\mathrm{b}}(\operatorname{End} N) \xrightarrow{\mathrm{VdB}} \mathbf{D}^{\mathrm{b}} X.$$

In what follows, we omit VdB from notation whenever convenient, thus for example writing Ψ_i for both the mutation equivalence and the flop functor in the above theorem.

By examining the connectedness of $\operatorname{ExQuiv} \operatorname{MMG}^N(R)$ we also conclude that any two birational models of X are connected by a chain of single-curve flops, i.e. given $W, W' \in \operatorname{Bir}(\frac{X}{Z})$ there is a $\mathbb{J}(W)$-path $\nu = \nu_{i_n} \dots \nu_{i_1}$ and a *sequence of flops*

$$W \dashrightarrow \nu_{i_1} W \dashrightarrow \nu_{i_2} \nu_{i_1} W \dashrightarrow \dots \dashrightarrow \nu_{i_n} \dots \nu_{i_1} W = W'.$$

Extending the notion of Bridgeland–Chen functors to chains of flops, in this case we say the *flop functor* associated to ν is the equivalence

$$\mathbf{D}^b \nu W \xrightarrow{\text{VdB}^{-1}} \mathbf{D}^b(\operatorname{End} \nu M) \xrightarrow{\Psi_\nu(-)=\mathbf{R}\mathrm{Hom}(\mathrm{Hom}_R(\nu M, M),-)} \mathbf{D}^b(\operatorname{End} M) \xrightarrow{\text{VdB}} \mathbf{D}^b W$$

which is evidently independent of the choice of ν. If we choose the sequence of flops ν to be *atomic* (i.e. atomic as a path in $\mathrm{MM}^N(R)$) then corollary 4.9 shows that the flop functor $\mathbf{D}^b \nu W \to \mathbf{D}^b W$ coincides with the composite $\Psi_{\nu_1} \circ \ldots \circ \Psi_{\nu_n}$ (where we omit VdB from the notation). It can be seen (lemma 4.16) that atomic sequences of flops are precisely those which have minimal length among all sequences of single-curve flops between the given birational models, in particular any there is at least one atomic sequence of flops between any two birational models.

§ 5.2 Flops and wall–crossing. Suppose W is a birational model of X with associated Dynkin data $\mathfrak{J} = \mathbb{J}(W)$. Given the flopping contraction $\pi : W \to Z$ with reduced exceptional fiber $\bigcup_{i \in \Delta \setminus \mathfrak{J}} C_i$, it is well known [see for example Van04, lemma 3.4.3] that the Picard group $\operatorname{Pic} W$ is naturally dual to the group of π-relative 1-cycles

$$Z_1(W) = \bigoplus_{i \in \Delta \setminus \mathfrak{J}} \mathbb{Z} \cdot C_i$$

via the intersection pairing $(\mathcal{L} \cdot C_i) = \deg(\mathcal{L}|_{C_i})$. In particular it is a free Abelian group of rank $|\Delta \setminus \mathfrak{J}|$, and if $\mathcal{V}(\frac{W}{Z})$ has indecomposable summands $(\mathcal{N}_i')_{i \in \Delta \setminus \mathfrak{J}}$ indexed as in lemma 3.8, then the line bundles $\mathcal{L}_i = (\det \mathcal{N}_i')^\vee$ $(i \in \Delta \setminus \mathfrak{J})$ form a basis of $\operatorname{Pic} W$ dual to the standard basis of $Z_1(W)$. Write $\operatorname{Pic}^+ W$ for the submonoid of $\operatorname{Pic} W$ generated by the basis elements, this is the monoid of *nef* line bundles (i.e. line bundles which intersect every exceptional curve non-negatively).

Now the birational map $\rho : W \dashrightarrow X$ is an isomorphism in codimension 1, and thus induces an isomorphism of Picard groups $\rho_* : \operatorname{Pic} W \to \operatorname{Pic} X$ given by taking proper transforms of divisors. This allows us to consider the submonoid $\rho_* \operatorname{Pic}^+ W \subset \operatorname{Pic} X$ of line bundles on X which are nef on W, and we now argue that considering all such submonoids gives a decomposition

$$\operatorname{Pic} X = \bigcup_{W \in \operatorname{Bir}(\frac{X}{Z})} \rho_* \operatorname{Pic}^+ W \tag{17}$$

whose combinatorics are given by the wall–crossing rules of Iyama–Wemyss.

For this it is convenient to consider the space of $\mathbb{R}$-divisors $\operatorname{Pic}_{\mathbb{R}} X = \operatorname{Pic} X \otimes \mathbb{R}$. The nef cone $\operatorname{Pic}^+_{\mathbb{R}} W$ is then an orthant generated by the basis vectors $\mathcal{L}_i$, and the isomorphisms of Picard groups given by proper transforms extend linearly to give isomorphisms $\rho_* \operatorname{Pic}_{\mathbb{R}} W \to \operatorname{Pic}_{\mathbb{R}} X$ so that the nef cone $\rho_* \operatorname{Pic}^+_{\mathbb{R}} W$ is a rational polyhedral cone in $\operatorname{Pic}_{\mathbb{R}} X$. We then show the following.

Proposition 5.5. *Given a flopping contraction $\pi : X \to Z$, the collection of rational polyhedral cones given by*

$$\operatorname{Mov}(X) = \bigcup_{W \in \operatorname{Bir}(\frac{X}{Z})} \operatorname{faces}(\rho_* \operatorname{Pic}^+_{\mathbb{R}} W)$$

forms a complete simplicial fan in $\operatorname{Pic}_{\mathbb{R}} X$ in which each nef cone $\rho_* \operatorname{Pic}^+_{\mathbb{R}} W$ is full-dimensional.

Further, the natural linear isomorphism of $\operatorname{Pic}_{\mathbb{R}} X$ with the hyperplane $\{\theta \in \Theta(\underline{\Delta} \setminus \mathfrak{J}) \mid \theta(\delta_{\mathfrak{J}}) = 0\}$ induces an isomorphism of fans $\operatorname{Mov}(X) \to \operatorname{Arr}(\Delta, \mathfrak{J})$ such that for each sequence of flops ν from X, the cone $\rho_* \operatorname{Pic}^+_{\mathbb{R}}(\nu X)$ gets identified with the chamber $\nu C^0_{\mathfrak{J}}$.

Note the decomposition (17) can be immediately deduced by noting that each nef monoid is the set of integral points in the corresponding nef cone.

In order to prove proposition 5.5, we first explain the construction of the natural map $\operatorname{Pic}_{\mathbb{R}} X \to \Theta(\underline{\Delta}, \mathfrak{J})$. Recall that the vector space $\Theta(\underline{\Delta}, \mathfrak{J})$ is dual to the restricted root lattice $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$, which we identify with the Grothendieck group $\mathbf{K}\Lambda$ and hence (across the equivalence (4)) with $\mathbf{K}X$. This identification is such that the sheaf $\mathcal{O}_{C_i}(-1)$

supported on the curve C_i ($i \in \Delta \setminus \mathfrak{J}$) has $\mathbf{K}$-theory class given by the restricted root $\alpha_i \in \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$, while the structure sheaf of the exceptional fiber has class $-\alpha_0$. The imaginary restricted root $\delta_{\mathfrak{J}}$, as we now show, corresponds precisely to the class of a skyscraper sheaf.

Lemma 5.6. *Let* $p \in X$ *be a closed point on the exceptional fiber of* $\pi : X \to Z$. *Identifying* $\mathbf{K}\,X$ *with* $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$ *as above, any skyscraper sheaf* $\mathcal{O}_p \in \mathcal{coh}\,X$ *has* $\mathbf{K}$*-theory class equal to* $\delta_{\mathfrak{J}}$.

Proof. By [Kar17, theorem 5.2.4], the $\mathbf{K}$-theory class of $\mathcal{O}_p$ is determined by the indecomposable summands of $\mathcal{N}$ as $[\mathcal{O}_p] = \sum \mathrm{rk}(\mathcal{N}_i) \cdot \alpha_i$. By [IW, proposition 9.4], this vector is precisely the imaginary root $\delta_{\mathfrak{J}}$. ☐

Now the objects in $\mathcal{coh}\,X$ have proper support of dimension $\leqslant 1$, so for any $\mathcal{L} \in \mathrm{Pic}\,X$ and $\mathcal{F} \in \mathcal{coh}\,X$ we have a well-defined Snapper–Kleiman intersection number

$$(\mathcal{L} \cdot \mathcal{F}) = \chi(\mathcal{F}) - \chi(\mathcal{L}^\vee \otimes \mathcal{F}).$$

By standard properties of the intersection product [see for example Kol96, §VI] this defines a $\mathbb{Z}$-bilinear map $(- \cdot -) : \mathrm{Pic}\,X \otimes \mathbf{K}\,X \to \mathbb{Z}$. The intersection pairings across all birational models are compatible with various identifications induced by flops.

Lemma 5.7. *Given a birational model* $W = \nu X$, *the two intersection pairings*

$$\mathrm{Pic}\,X \otimes \mathbf{K}\,X \to \mathbb{Z} \quad \textit{and} \quad \mathrm{Pic}\,W \otimes \mathbf{K}\,W \to \mathbb{Z}$$

are compatible with the isomorphisms $\mathrm{Pic}\,W \cong \mathrm{Pic}\,X$ *and* $\mathbf{K}\,W \cong \mathbf{K}\,X$, *where the Picard groups are identified by taking proper transforms across* $\rho : W \dashrightarrow X$, *and the Grothendieck groups are identified via the flop equivalence.*

Proof. It suffices to prove this for a single-curve flop $W = \nu_i X$, $i \in \Delta \setminus \mathfrak{J}$.

Note both $\mathrm{Pic}\,X$ and $\mathrm{Pic}\,W$ are equipped with a preferred choice of basis– continuing to write $\mathcal{N}_j$ ($j \in \underline{\Delta} \setminus \mathfrak{J}$) for the summands of $\mathcal{N} = \mathcal{V}(\frac{X}{Z})$, $\mathrm{Pic}\,X$ is generated by the line bundles $(\det \mathcal{N}_j)^\vee = \mathcal{L}_j$. Likewise writing $\mathcal{N}'_j$ ($j \in \underline{\Delta} \setminus \nu_i\mathfrak{J}$) for the summands of $\mathcal{N}' = \mathcal{V}(\frac{W}{Z})$, we have the generators $(\det \mathcal{N}'_j)^\vee = \mathcal{L}'_j$ of $\mathrm{Pic}\,W$.

The Grothendieck groups too have preferred choices of bases in which the intersection product is readily computed– for $\mathbf{K}\,W$ we choose the class of a skyscraper $\mathcal{O}_p$ ($p \in W$) and the classes of simple perverse sheaves $S_k = \mathcal{O}_{C_k}(-1)$ ($k \in \Delta \setminus \nu_i\mathfrak{J}$), and likewise for $\mathbf{K}\,X$.

To prove the result it thus suffices to show the equalities of intersection pairings

$$(\rho_*\mathcal{L}'_j, \Psi_i\mathcal{O}_p) = (\mathcal{L}'_j, \mathcal{O}_p) \quad \text{and} \quad (\rho_*\mathcal{L}'_j, \Psi_i S_k) = (\mathcal{L}'_j, S_k) \qquad \text{for all } j,k \in \Delta \setminus \nu_i\mathfrak{J}.$$

The first equality clearly holds since both intersections are zero (the object $\Psi_i\mathcal{O}_p$ has the same class as a skyscraper sheaf on X). The latter equalities, on the other hand, can be shown explicitly by expressing $\rho_*\mathcal{L}'_j$ and $\Psi_i S_k$ in terms of the standard bases of $\mathrm{Pic}\,X$ and $\mathbf{K}\,X$ respectively and computing both sides of the expression. We explain how to do this.

To compute the proper transform of $\mathcal{L}'_j$, we first note that the birational maps $\pi : X \to Z$, $\pi : W \to Z$, and $\rho : W \dashrightarrow X$ are all isomorphisms away from a codimension two locus and hence induce equivalences between categories of reflexive sheaves. Thus we can compare line bundles on the two varieties by taking pushforwards to Z, where theorem 5.4 shows $\pi_*\mathcal{V}(\frac{X}{Z}) = \nu_{\iota(i)}(\pi_*\mathcal{V}(\frac{X}{Z}))$. Thus for $j \in \underline{\Delta} \setminus (\mathfrak{J} + i)$ we have

$$\pi_*\mathcal{N}'_j = \pi_*\mathcal{N}_j = N_j, \quad \text{so that} \quad \rho_*\mathcal{L}'_j = \mathcal{L}_j.$$

On the other hand, $\pi_*\mathcal{N}_i = N_i$ is the kernel of a minimal right $\mathrm{add}(\nu_i N/N_{\iota(i)})$ approximation f of $\pi_*\mathcal{N}'_{\iota(i)} = N_{\iota(i)}$. Further, $\nu_i N/N_{\iota(i)}$ contains $N_0 = R$ as a summand, whence the approximation f is surjective i.e. we have an exact sequence

$$0 \to N_i \xrightarrow{\ g\ } \bigoplus_{j \in \Delta \setminus (\mathfrak{J}+i)} N_j^{\oplus b_j} \xrightarrow{\ f\ } N_{\iota(i)} \to 0 \tag{18}$$

for some tuple of non-negative integers (b_j). Considering determinants gives

$$\det N_{\iota(i)} = \bigotimes_{j\in\underline{\Delta}\setminus(\mathfrak{J}+i)} (\det N_j)^{\otimes b_j} \otimes (\det N_i)^{\vee}, \quad \text{so that} \quad \rho_* \mathcal{L}'_{\iota(i)} = \bigotimes_{j\in\underline{\Delta}\setminus(\mathfrak{J}+i)} \mathcal{L}_j^{\otimes b_j} \otimes \mathcal{L}_i^{\vee}.$$

Thus we have computed the proper transforms of all generators of $\operatorname{Pic} W$. Moreover, [IW14, proposition 6.4 (1)] shows the map g in the exchange sequence eq. (18) is a minimal $\operatorname{add}(N/N_i)$-approximation of N_i. This allows us to use lemmas 4.3 and 4.5 and compute the classes $[\Psi_i S_k] \in \mathbf{K}\, X$ as

$$[\Psi_i S_{\iota(i)}] = -[S_i], \qquad [\Psi_i S_k] = [S_k] + b_k [S_i] \quad (k \in \Delta \setminus (\mathfrak{J}+i)).$$

The equalities $(\rho_* \mathcal{L}'_j, \Psi_i S_k) = (\mathcal{L}'_j, S_k)$ are then straightforward to verify. □

The intersection product $\operatorname{Pic} X \otimes \mathbf{K}\, X \to \mathbb{R}$ naturally gives a linear map $\operatorname{Pic}_{\mathbb{R}} X \to \operatorname{Hom}(\mathbf{K}\, X, \mathbb{R}) \cong \Theta(\underline{\Delta} \setminus \mathfrak{J})$. By showing this map is an isomorphism onto the hyperplane $\{\delta_{\mathfrak{J}} = 0\}$ and tracking various nef cones across it, we can now prove the proposition.

Proof of proposition 5.5. The intersection of any line bundle with a sheaf that has zero-dimensional support is trivial, thus considering $\mathcal{F} = \mathcal{O}_p$ (the skyscraper sheaf at a closed point) and using lemma 5.6 shows that the map $\operatorname{Pic}_{\mathbb{R}} X \to \Theta(\underline{\Delta} \setminus \mathfrak{J})$ constructed above has image in $\{\delta_{\mathfrak{J}} = 0\}$. On the other hand considering the sheaves $\mathcal{F} = \mathcal{O}_{C_i}(-1)$ shows the map is injective, and hence an isomorphism onto the hyperplane $\{\delta_{\mathfrak{J}} = 0\}$ by comparing dimensions.

Now the nef cone $\operatorname{Pic}^+_{\mathbb{R}} X$ is spanned by divisors which pair with each $\alpha_i = [\mathcal{O}_{C_i}(-1)]$ non-negatively, and hence gets identified with $C^0_{\mathfrak{J}}$ by the above injection. If $W = \nu X$ is another birational model, its nef cone likewise gets identified with the chamber $C^0_{\nu\mathfrak{J}} \subset \Theta(\underline{\Delta}, \nu\mathfrak{J})$. By lemma 5.7 these identifications are compatible with flops and mutation, i.e. there is a commutative square

$$\begin{array}{ccc} \operatorname{Pic}_{\mathbb{R}} W & \xrightarrow{\rho_*} & \operatorname{Pic}_{\mathbb{R}} X \\ \downarrow & & \downarrow \\ \Theta(\underline{\Delta} \setminus \nu\mathfrak{J}) & \xleftarrow{\varphi_\nu^\vee} & \Theta(\underline{\Delta} \setminus \mathfrak{J}). \end{array}$$

Thus the nef cone $\rho_* \operatorname{Pic}^+_{\mathbb{R}} W$ in $\operatorname{Pic}_{\mathbb{R}} X$ gets mapped to the cone $(\varphi_\nu^\vee)^{-1} C^0_{\nu\mathfrak{J}} = \nu C^0_{\mathfrak{J}}$ in $\Theta(\underline{\Delta}, \mathfrak{J})$, in particular each nef cone is full dimensional and all maximal cones in $\operatorname{Mov}(X)$ arise as nef cones. It follows that there is an isomorphism of fans $\operatorname{Mov}(X) \to \operatorname{Arr}(\Delta, \mathfrak{J})$ of the required form, and the remaining properties (completeness, simpliciality) of $\operatorname{Mov}(X)$ can be deduced from the corresponding properties of $\operatorname{Arr}(\Delta, \mathfrak{J})$. □

We now examine geometric consequences. The completeness of the fan $\operatorname{Mov}(X)$ implies that any line bundle $\mathcal{L} \in \operatorname{Pic} X$, after some appropriate sequence of flops, becomes nef and hence globally generated [see for example Van04, lemma 3.4.5]. In particular the base locus of the linear system associated to $\mathcal{L}$ is contained entirely within the exceptional curves in X, which has co-dimension 2. Recalling that an $\mathbb{R}$-divisor is *movable* if and only if it is a non-negative linear combinations of such line bundles, we have the following.

Corollary 5.8. *If $\pi : X \to Z$ is a flopping contraction, every divisor on X is movable.*

This explains the notation $\operatorname{Mov}(X)$ for the fan, since its support can be identified with the cone of movable divisors on X. When X is a minimal model of Z, work of Kawamata [Kaw88], Kollár [Kol89], Mori [Mor82], and Reid [Rei] shows that the (closure of the) cone of movable divisors can be decomposed into a union of nef cones of minimal models in a way that that the nef cones of non-isomorphic models have disjoint interiors [the result is summarised in Mat02, theorem 12.2.7]. Via mutation combinatorics of birational models, modifying generators, and chambers in the intersection arrangement, this decomposition theorem generalises to arbitrary flopping contractions over isolated cDV singularities.

Corollary 5.9. *Given a flopping contraction $\pi : X \to Z$, the nef cones $\rho_* \operatorname{Pic}^+_{\mathbb{R}} W$ for $W \in \operatorname{Bir}(\frac{X}{Z})$ cover the cone of movable divisors and their interiors (i.e. the corresponding ample cones) are pairwise disjoint.*

Proof. The only claim that is unproven so far is that non-isomorphic birational models have disjoint ample cones, equivalently (since $\operatorname{Mov}(X)$ is a fan) that non-isomorphic birational models cannot have the same nef cones. But if νX and $\nu' X$ have the same nef cone, the correspondence of proposition 5.5 shows we have $\nu C^0_{\mathfrak{J}} = \nu' C^0_{\mathfrak{J}}$ in $\operatorname{Cham}(\Delta, \mathfrak{J})$. By the general affine Auslander–McKay correspondence [IW, theorem 0.18], the assignment $\nu C^0_{\mathfrak{J}} \mapsto \nu N$ is a bijection $\operatorname{Cham}(\Delta, \mathfrak{J}) \to \operatorname{MMG}^N R$ and thus we have an equality of modifying generators $\nu N = \nu' N$. But by theorem 5.3, this is not possible unless $\nu X \cong \nu' X$ as required. □

In particular, the assignment to nef cones gives an isomorphism $\operatorname{Bir}(\frac{X}{Z}) \to \operatorname{Cham}(\Delta, \mathfrak{J})$ of sets with Δ-mutation. Karmazyn [Kar17, §5.2] and Wemyss [Wem18, §5] use geometric invariant theory to provide a natural interpretation of the inverse map, which we briefly outline next.

By [Wem18, theorem 2.15] the category $\mathcal{P}er(\frac{X}{Z}) \cong \mathcal{mod}\Lambda$ can be identified with a subcategory of representations of a quiver with vertices $\underline{\Delta} \setminus \mathfrak{J}$. Viewing elements of $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J}) = \bigoplus_{i \in \Delta \setminus \mathfrak{J}} \mathbb{Z} \cdot \alpha_i$ as dimension vectors for this quiver, the identification is such that the dimension vector of the representation corresponding to $x \in \mathcal{per}(\frac{X}{Z})$ is precisely its $\mathbf{K}$-theory class.

Under the above identifications, elements of the dual vector space $\Theta(\underline{\Delta} \setminus \mathfrak{J})$ naturally give stability parameters (in the sense of [Kin94]) on $\mathcal{P}er(\frac{X}{Z})$ and thus for any stability parameter $\theta \in \{\delta_{\mathfrak{J}} = 0\}$ we can construct the coarse moduli space $\mathcal{M}(\theta, \delta_{\mathfrak{J}})$ of θ-semistable perverse sheaves with dimension vector $\delta_{\mathfrak{J}}$. When θ is generic in a chamber $\nu C^0_{\mathfrak{J}}$, there are no strictly semistable objects and it is shown that there is an isomorphism $\mathcal{M}(\theta, \delta_{\mathfrak{J}}) \cong \nu X$ in a way that the closed point $p \in \nu X$ corresponds to the 'perverse point sheaf' $\Psi_\nu \mathcal{O}_p \in \mathcal{P}er(\frac{X}{Z})$. This moduli construction defines the inverse map $\operatorname{Cham}(\Delta, \mathfrak{J}) \to \operatorname{Bir}(\frac{X}{Z})$.

The upshot for us is that this gives excellent control over the subcategories of semistable objects in $\mathcal{per}(\frac{X}{Z})$.

Proposition 5.10. *For a chamber $\nu C^0_{\mathfrak{J}} \in \operatorname{Cham}(\Delta, \mathfrak{J})$ and a generic vector $\theta \in \nu C^0_{\mathfrak{J}}$, an object of $H = \mathcal{per}(\frac{X}{Z})$ is θ-stable if and only if it is equal to a perverse point sheaf $\Psi_\nu \mathcal{O}_p$ for some closed point p on the exceptional fiber of $\pi : \nu X \to Z$. In particular, subcategory of semistable objects associated to $\nu C^0_{\mathfrak{J}}$ is given by the extension closure of such objects, i.e.*

$$H_{ss}\left(\nu C^0_{\mathfrak{J}}\right) = \left\langle \Psi_\nu \mathcal{O}_p \mid p \in \pi^{-1}[\mathfrak{m}] \subset \nu X \right\rangle .$$

Proof. By [Wem18, theorem 5.12], the flop functor $\Psi_\nu : \mathbf{D}^b(\nu X) \to \mathbf{D}^b X$ restricts to an equivalence of categories

$$\left\{x \in \mathcal{P}er(\tfrac{\nu X}{Z}) \mid x \text{ is } \varphi_\nu^{\vee}\theta\text{-semistable}\right\} \longrightarrow \left\{x \in \mathcal{P}er(\tfrac{X}{Z}) \mid x \text{ is } \theta\text{-semistable}\right\}$$

and therefore an object of $\mathcal{per}(\frac{X}{Z})$ is θ-semistable if and only if it is of the form $\Psi_\nu(x)$ for some $\varphi_\nu^{\vee}\theta$-semistable object $x \in \mathcal{per}(\frac{\nu X}{Z})$. Thus it suffices to show that for a generic $\theta \in C^0_{\mathfrak{J}}$, the θ-stable objects in $\mathcal{per}(\frac{X}{Z})$ are precisely all the skyscraper sheaves on the exceptional fiber. Given this, the description of $H_{ss}\left(\nu C^0_{\mathfrak{J}}\right)$ follows by recalling that any semistable object admits a finite filtration by stable ones.

Now if an object is θ-stable, it remains so upon small perturbations of θ within the hyperplane $\{\delta_{\mathfrak{J}} = 0\}$ and thus it must have dimension vector $n\delta_{\mathfrak{J}}$ for some integer $n > 0$. The objects with $n = 1$ can be read off from the closed points in Karmazyn's moduli space $\mathcal{M}(\theta, \delta_{\mathfrak{J}}) \cong X$, thus the collection of θ-stable objects in $\mathcal{per}(\frac{X}{Z})$ that have $\mathbf{K}$-theory class $\delta_{\mathfrak{J}}$ is precisely

$$\left\{\mathcal{O}_p \mid p \in \pi^{-1}[\mathfrak{m}] \subset X\right\} .$$

We show the case $n \geqslant 2$ does not occur, by replicating the argument of [Gar22, lemma 4.12]. Indeed suppose for the sake of contradiction that $x \in \mathcal{per}(\frac{X}{Z})$ is θ-stable and $[x] = n\delta_{\mathfrak{J}}$ for $n \geqslant 2$. Since any skyscraper sheaf $\mathcal{O}_p \in \mathcal{P}er(\frac{X}{Z})$ is θ-stable and necessarily distinct from x, we have $\operatorname{Hom}(x, \mathcal{O}_p) = 0$ for all $p \in X$. Combining this with the fact that x is a two-term complex of coherent sheaves, we see that the sheaf $H^0_{\mathcal{coh} X}(x)$ has empty support and hence $x[-1]$ is a sheaf on X. Choosing a sufficiently ample bundle $\mathcal{L}$ on X then gives us $\chi(\mathcal{L} \otimes x[-1]) \geqslant 0$, but this is absurd since x is numerically equivalent to some skyscraper sheaf $\mathcal{O}_p^{\oplus n}$. □

§ 5.3 Classic, mixed, and reversed geometric hearts. We use the geometries of flops to construct a plethora of t-structures on $\mathbf{D}^0 X$. For instance, the natural heart $\mathcal{C}oh\, W \subset \mathbf{D}^b W$ restricts to give a Noetherian t-structure $coh\, W \subset \mathbf{D}^0 W$, which is the full subcategory of coherent sheaves supported on the exceptional fiber of $\pi : W \to Z$. Now as in [BPPW24, example 2.6] we can consider the torsion class of sheaves with zero-dimensional support, giving a torsion pair

$$coh\, X = \langle \mathcal{O}_p \mid p \in \pi^{-1}[\mathfrak{m}] \rangle \, * \, coh^\circ X$$

where $coh^\circ X$ is the corresponding torsion-free class of pure sheaves with one dimensional support in $\pi^{-1}[\mathfrak{m}]$. The corresponding tilt is an Artinian category which we call the *reversed geometric heart*, given by

$$\overline{coh}\, X = coh^\circ X \, * \, \langle \mathcal{O}_p[-1] \mid p \in \pi^{-1}[\mathfrak{m}] \rangle.$$

Evidently, we have $\overline{coh}\, X < coh\, X$ in t-str$(\mathbf{D}^0 X)$. Further since the torsion class is generated by simple objects, we can immediately deduce the following description of the interval $[\overline{coh}\, X, coh\, X]$.

Lemma 5.11. *Let* $K \in$ t-str$(\mathbf{D}^0 X)$ *be the heart of a t-structure satisfying* $\overline{coh}\, X \leqslant K \leqslant coh\, X$. *Then there is a subset of the exceptional fiber* $Q \subseteq \pi^{-1}[\mathfrak{m}]$ *such that* K *is the tilt of* $coh\, X$ *in the torsion class of sheaves with zero-dimensional support in* Q*, i.e.*

$$K = \langle \mathcal{O}_p \mid p \in \pi^{-1}[\mathfrak{m}] \setminus Q \rangle \, * \, coh^\circ X \, * \, \langle \mathcal{O}_p[-1] \mid p \in Q \rangle. \tag{19}$$

Further every heart in the interval $[\overline{coh}\, X, coh\, X] \subset$ t-str$(\mathbf{D}^0 X)$ *arises in this way, giving an isomorphism of posets between the said interval and the boolean lattice on the closed points of* $\pi^{-1}[\mathfrak{m}]$.

Thus we have described geometric hearts of three flavours– classic ($coh\, X$), reversed ($\overline{coh}\, X$), and *mixed* (i.e. everything in between). Given any other birational model $W = \nu X$, we can likewise construct the interval $[\overline{coh}\, W, coh\, W] \subset \mathbf{D}^0 W$ and track it across the flop functor Ψ_ν to obtain an interval of geometric t-structures

$$\Psi_\nu[\overline{coh}\, W, coh\, W] = \left\{ \Psi_\nu K \mid \overline{coh}\, W \leqslant K \leqslant coh\, W \text{ in t-str}(\mathbf{D}^0 W) \right\} \quad \subset \text{t-str}(\mathbf{D}^0 X).$$

Remark 5.12. Geometric hearts and their associated torsion theories furnish a rich bank of examples when studying chain conditions on Abelian categories and wide generation of torsion theories. In particular, the table in § 2.3 is constructed using the following observations.

(When does a geometric heart arise from semibricks?) Suppose the torsion pair $H = T * F$ is such that the tilt $K = F * T[-1]$ lies in $[\overline{coh}\, X, coh\, X]$, and is associated to $Q \subseteq \pi^{-1}[\mathfrak{m}]$ as in lemma 5.11. Evidently each $\mathcal{O}_p$ ($p \in \pi^{-1}[\mathfrak{m}] \setminus Q$) and each $\mathcal{O}_p[-1]$ ($p \in Q$) is simple in K. Considering the decomposition (19) shows that any simple object of K must be of this form; in particular a sheaf $k \in coh^\circ X$ cannot be simple in K since the sets $\mathrm{Hom}(\mathcal{O}_p[-1], k)$ and $\mathrm{Hom}(k, \mathcal{O}_p)$ are both non-zero for any closed point $p \in \mathrm{Supp}(k)$.

By a similar reasoning, we see that every non-zero object in K has a simple sub-object (i.e. T is widely generated, see proposition 2.6) if and only if Q contains at least one point in each exceptional curve $C_i \subset X$. Likewise, non-zero object in K has a simple quotient (i.e. F is widely generated) if and only if the complement $\pi^{-1}[\mathfrak{m}] \setminus Q$ contains at least one point in every exceptional curve $C_i \subset X$.

(What chain conditions do geometric hearts satisfy?) It is a classical fact that $coh\, X$ is a Noetherian category. On the other hand, any geometric heart $K \in [\overline{coh}\, X, coh\, X)$ is necessarily non-Noetherian, since K then contains some $\mathcal{O}_p[-1]$ ($p \in C_i \subset X$) and hence the sequence of morphisms $\mathcal{O}_{C_i} \to \mathcal{O}_{C_i}(1) \to \mathcal{O}_{C_i}(2) \to \ldots$ (each of which has cone $\mathcal{O}_p \in K[1]$) gives an infinite chain of proper surjections in K.

Dually, the heart $\overline{coh}\, X$ is the only Artinian heart in the interval $[\overline{coh}\, X, coh\, X]$. Indeed if a morphism $x \to y$ is injective in $\overline{coh}\, X$ with non-zero cokernel y/x, then considering cohomologies with respect to $coh\, X$ produces a long exact sequence

$$0 \to \underbrace{\mathrm{H}^0(x) \to \mathrm{H}^0(y) \to \mathrm{H}^0(y/x)}_{\text{pure sheaves with one-dimensional support}} \longrightarrow \underbrace{\mathrm{H}^1(x) \to \mathrm{H}^1(y) \to \mathrm{H}^1(y/x)}_{\text{sheaves with zero-dimensional support}} \to 0$$

in $\mathcal{coh}\,X$. In particular, either H^0x has lower rank than H^0y on some exceptional curve $C_i \subset X$, or y/x has zero-dimensional support (i.e. $H^0(y/x) = 0$ and the length of H^1x is lower than that of H^1y. Thus $\overline{\mathcal{coh}}\,X$ has no infinite descending chains of inclusions ... $\hookrightarrow x_2 \hookrightarrow x_1 \hookrightarrow x$.

Intermediacy of geometric hearts. We now analyse geometric hearts on different birational models $W \in \mathrm{Bir}\left(\frac{X}{Z}\right)$ in relation to the reference heart $H = \mathcal{per}\left(\frac{X}{Z}\right)$ in $\mathbf{D}^0X$, and prove the following.

Theorem 5.13. *For ν a sequence of flops from X and $W = \nu X$ the corresponding birational model, the translate of any t-structure in $[\overline{\mathcal{coh}}\,W, \mathcal{coh}\,W]$ across the flop functor Ψ_ν is intermediate with respect to H. Moreover the t-structures $\Psi_\nu\,\overline{\mathcal{coh}}\,W$ and $\Psi_\nu\,\mathcal{coh}\,W$ are determined numerically as*

$$\Psi_\nu\,\overline{\mathcal{coh}}\,W = H_{tt}\left(\nu C^0_{\mathfrak{J}}\right), \qquad \Psi_\nu\,\mathcal{coh}\,W = H^{tt}\left(\nu C^0_{\mathfrak{J}}\right),$$

and thus an intermediate heart $K \in \mathrm{tilt}(H)$ has heart cone $\mathbf{C}\,K = \nu C^0_{\mathfrak{J}}$ if and only if it lies in $\Psi_\nu[\overline{\mathcal{coh}}\,W, \mathcal{coh}\,W]$.

Combining this with lemma 5.11 immediately yields the following.

Corollary 5.14. *For any spherical $\mathfrak{J}$-path ν and chamber $\sigma = \nu C^0_{\mathfrak{J}}$, the interval $[H_{tt}\sigma, H^{tt}\sigma]$ in $\mathrm{tilt}(H)$ is isomorphic to the boolean lattice of closed points on the (reduced) exceptional fiber C of $\pi : \nu X \to Z$. Further, the brick labels arising in this interval are precisely the perverse point sheaves $\Psi_\nu \mathcal{O}_p$ for $p \in C$.*

We now prove theorem 5.13. When the sequence of flops ν is trivial, this is straightforward.

Proof of theorem 5.13 when $W = X$. Recall that the tilting bundle $\mathcal{V}\left(\frac{X}{Z}\right)$ allows for the description (5) of $\mathcal{Per}\left(\frac{X}{Z}\right)$ as a subcategory of $\mathcal{Coh}\,X[0,1]$. Accordingly in $\mathbf{D}^0X$ we have $\mathcal{coh}\,X \subset \mathcal{per}\left(\frac{X}{Z}\right)[-1,0]$, i.e. $\mathcal{coh}\,X$ lies in $\mathrm{tilt}(H)$.

Now considering the collection of objects $\{\mathcal{O}_{C_i}(n) \mid i \in \Delta \setminus \mathfrak{J},\quad n \in \mathbb{Z}\}$ shows that the heart cone $\mathbf{C}(\mathcal{coh}\,X)$ must be contained in the cone $C^0_{\mathfrak{J}}$. Thus it follows from the discussion in § 5.2 that every functional in $\mathbf{C}(\mathcal{coh}\,X)$ can be expressed on classes of sheaves as

$$[\mathcal{F}] \mapsto \sum_{i \in \Delta\setminus\mathfrak{J}} a_i(\mathcal{L}_i \cdot \mathcal{F})$$

for some tuple of real numbers $a_i \geqslant 0$. But conversely every such tuple yields a functional that is non-negative on objects in $\mathcal{coh}\,X$, i.e. we have $\mathbf{C}(\mathcal{coh}\,X) = C^0_{\mathfrak{J}}$.

Picking a generic $\theta \in C^0_{\mathfrak{J}}$, it follows from lemma 2.13 that $\mathcal{coh}\,X$ is the tilt of the numerically defined category $H^{tt}(\theta)$ in the torsion class $U = \mathcal{coh}\,X[1] \cap H_{ss}(\theta)$. But every θ-semistable object is an extension of skyscraper sheaves (proposition 5.10), so we must have in fact have $U = 0$ i.e. $H^{tt}(\theta) = \mathcal{coh}\,X$, and consequently $H_{tt}(\theta) = \overline{\mathcal{coh}}\,X$ as claimed. □

When a non-trivial sequence of flops ν is involved, the intermediacy of $\Psi_\nu\,\mathcal{coh}\,W$ with respect to $H = \mathcal{per}\left(\frac{X}{Z}\right)$ is not immediate from constructions. The idea, rather, is to consider the *interval* $\left(\mathcal{coh}\,W, \mathcal{per}\left(\frac{W}{Z}\right)\right] \subset \text{t-str}(\mathbf{D}^0W)$ and show that its track under the flop functor lies entirely in $\mathrm{tilt}(H)$. Since $\mathrm{tilt}(H)$ is a complete lattice, it then follows that

$$\Psi_\nu(\mathcal{coh}\,W) = \inf\ \Psi_\nu\left(\mathcal{coh}\,W, \mathcal{per}\left(\tfrac{W}{Z}\right)\right] \quad \in \mathrm{tilt}(H).$$

Given the intermediacy of $\Psi_\nu\,\mathcal{coh}\,W$, the statements about the numerical tilts $H_{tt}(\nu C^0_{\mathfrak{J}}), H^{tt}(\nu C^0_{\mathfrak{J}})$ follow by examining the semistable objects in $H_{ss}(\nu C^0_{\mathfrak{J}})$ just as in the $W = X$ case.

Thus we examine the interval $\left(\mathcal{coh}\,X, \mathcal{per}\left(\frac{X}{Z}\right)\right]$, showing that it can in fact be defined $\mathbf{K}$-theoretically.

Lemma 5.15. *Given an algebraic heart $K \in \text{t-str}^{\mathrm{alg}}(\mathbf{D}^0X)$, the following are equivalent.*

(1) We have $\mathcal{coh}\,X \leqslant K \leqslant \mathcal{per}\left(\frac{X}{Z}\right)$ in the partial order on t-structures.

(2) The heart K contains the collection of objects $\mathcal{S} = \{\mathcal{O}_{C_i}(-1) \mid i \in \Delta \setminus \mathfrak{J}\} \cup \{\mathcal{O}_p \mid p$ a closed point of $\pi^{-1}[\mathfrak{m}]\}$.

(3) *The heart cone* $\mathbf{C}\,\mathsf{K} \subset \Theta(\underline{\Delta} \setminus \mathfrak{J})$ *lies in the region* $\{\delta_{\mathfrak{J}} > 0\} \cap \bigcap_{i \in \Delta \setminus \mathfrak{J}} \{\alpha_i \geqslant 0\}$.

Proof. The implications (1)⇒(2)⇒(3) are immediate by noting that any heart K satisfying $\mathcal{coh}\,X \leqslant \mathsf{K} \leqslant \mathcal{per}(\frac{X}{Z})$ must contain the subcategory $\mathcal{coh}\,X \cap \mathcal{per}(\frac{X}{Z}) \supset \mathcal{S}$, and in this case every $\theta \in \mathbf{C}\,\mathsf{K}$ must be non-negative on the classes of objects in $\mathcal{S}$ (we note that $\theta(\delta_{\mathfrak{J}})$ is non-zero whenever θ is in the heart cone of an algebraic heart).

Now if K is an algebraic heart satisfying (3), then $\mathbf{C}\,\mathsf{K}$ is non-zero and a generic parameter $\theta \in \mathbf{C}\,\mathsf{K}$ can be written as $\theta = \theta^0 + t \cdot \delta^*_{\mathfrak{J}}$ for some $\theta^0 \in C^0_{\mathfrak{J}}$, a real number $t > 0$, and the vector $\delta^*_{\mathfrak{J}} \in \Theta(\underline{\Delta} \setminus \mathfrak{J})$ defined by

(20) $$\delta^*_{\mathfrak{J}}(\delta_{\mathfrak{J}}) = 1, \quad \delta^*_{\mathfrak{J}}(\alpha_i) = 0 \quad \text{for all } i \in \Delta \setminus \mathfrak{J}.$$

Note that $\delta^*_{\mathfrak{J}}(\alpha_0) = 1$ so that $\delta^*_{\mathfrak{J}}[h] \geqslant 0$ for all $h \in \mathsf{H}$. It follows that we have

$$\begin{aligned} \mathsf{K}[1] \cap \mathsf{H} = \mathsf{H}_{\mathrm{tr}}(\theta) &= \{h \in \mathsf{H} \mid \theta[h'] < 0 \text{ for all non-zero factors } h \twoheadrightarrow h' \neq 0\} \\ &= \{h \in \mathsf{H} \mid \theta^0[h'] < -t \cdot \delta^*_{\mathfrak{J}}[h'] \text{ for all non-zero factors } h \twoheadrightarrow h' \neq 0\} \\ &\subseteq \{h \in \mathsf{H} \mid \theta^0[h'] < 0 \text{ for all non-zero factors } h \twoheadrightarrow h' \neq 0\} \\ &= \mathsf{H}_{\mathrm{tr}}(\theta^0) = \mathsf{H}^{\mathrm{tt}}(\theta^0)[1] \cap \mathsf{H}. \end{aligned}$$

But we have shown that $\mathsf{H}^{\mathrm{tt}}(\theta^0)$ coincides with $\mathcal{coh}\,X$, thus there is a containment $\mathsf{K}[1] \cap \mathsf{H} \subseteq \mathcal{coh}\,X[1] \cap \mathsf{H}$ which gives the relation (1) on the corresponding tilts. □

Write $[\mathcal{coh}\,X, \mathsf{H}]_{\mathrm{alg}}$ for the set of algebraic t-structures K satisfying $\mathcal{coh}\,X \leqslant \mathsf{K} \leqslant \mathsf{H}$, this is equal to $[\mathcal{coh}\,X, \mathsf{H}] \cap \mathrm{tilt}^+(\mathsf{H})$. We now argue that this poset captures all necessary information required to track geometric hearts under mutations.

Lemma 5.16. *Algebraic hearts are dense in the interval* $[\mathcal{coh}\,X, \mathsf{H}]$*, i.e. every heart* K *satisfying* $\mathcal{coh}\,X \leqslant \mathsf{K} \leqslant \mathsf{H}$ *is the infimimum (in the complete lattice* $\mathrm{tilt}(\mathsf{H})$*) of a subset of* $[\mathcal{coh}\,X, \mathsf{H}]_{\mathrm{alg}}$.

Proof. First observe that it suffices to prove that $\mathcal{coh}\,X$ is the infimum of $[\mathcal{coh}\,X, \mathsf{H}]_{\mathrm{alg}}$. Indeed if $\mathcal{coh}\,X \leqslant \mathsf{K} \leqslant \mathsf{H}$ is any other heart, then we can consider the poset

$$\left\{\inf\{\mathsf{K}, \mathsf{K}'\} \mid \mathsf{K}' \in [\mathcal{coh}\,X, \mathsf{H}]_{\mathrm{alg}}\right\}$$

which has infimum $\inf\{\mathsf{K}, \mathcal{coh}\,X\} = \mathsf{K}$, and further is contained in $[\mathcal{coh}\,X, \mathsf{H}]_{\mathrm{alg}}$ since every heart $\inf\{\mathsf{K}, \mathsf{K}'\}$ is contained in the interval $[\mathsf{K}', \mathsf{H}] \subset \mathrm{tilt}^+(\mathsf{H})$ and is therefore algebraic by theorem 4.1 (2).

Thus we show that $\mathcal{coh}\,X = \inf[\mathcal{coh}\,X, \mathsf{H}]_{\mathrm{alg}}$, or equivalently that we have the equality of torsion free classes

$$\mathcal{coh}\,X \cap \mathsf{H} = \bigcap_{\mathsf{K} \in [\mathcal{coh}\,X, \mathsf{H}]_{\mathrm{alg}}} \mathsf{K} \cap \mathsf{H}.$$

The containment $\subseteq$ is obvious, so we suppose we have an object h that lies in $\mathsf{K} \cap \mathsf{H}$ for every $\mathsf{K} \in [\mathcal{coh}\,X, \mathsf{H}]_{\mathrm{alg}}$ and show that we must have $h \in \mathcal{coh}\,X \cap \mathsf{H}$. Fix a generic $\theta^0 \in C^0_{\mathfrak{J}}$ so that $\mathcal{coh}\,X \cap \mathsf{H} = \mathsf{H}^{\mathrm{tf}}(\theta^0)$, and consider the parameter $\theta = \theta^0 + t \cdot \delta^*_{\mathfrak{J}}$ for $t > 0$. By lemma 5.15 θ lies in the heart cone $\mathbf{C}\,\mathsf{K}$ for some $\mathsf{K} \in [\mathcal{coh}\,X, \mathsf{H}]_{\mathrm{alg}}$, so we have $h \in \mathsf{K} \cap \mathsf{H} = \mathsf{H}^{\mathrm{tf}}(\theta)$, and hence $\theta[h'] \geqslant 0$ for all sub-objects $h' \hookrightarrow h$. Now we must have $0 \leqslant \delta^*_{\mathfrak{J}}[h'] \leqslant \delta^*_{\mathfrak{J}}[h]$ whenever h' is a sub-object of h, so it follows that $\theta^0[h'] \geqslant -t \cdot \delta^*_{\mathfrak{J}}[h]$ for all $h' \hookrightarrow h$. But since $t > 0$ was arbitrary we deduce $\theta^0[h'] \geqslant 0$ for all such h', i.e. $h \in \mathsf{H}^{\mathrm{tf}}(\theta^0)$ as required. □

The above results apply to any birational model $W = \nu X$, in particular algebraic hearts are dense in the interval $\left[\mathcal{coh}\,W, \mathcal{per}(\frac{W}{Z})\right]$ and an algebraic heart $\mathsf{K} \in \text{t-str}^{\mathrm{alg}}(\mathbf{D}^0 W)$ satisfies $\mathcal{coh}\,W \leqslant \mathsf{K} \leqslant \mathcal{per}(\frac{W}{Z})$ if and only if its heart cone lies in the region

(21) $$\{\delta_{\nu\mathfrak{J}} > 0\} \cap \bigcap_{i \in \Delta \setminus \nu\mathfrak{J}} \{\alpha_i \geqslant 0\} \quad \subset \Theta(\underline{\Delta} \setminus \nu\mathfrak{J}).$$

In particular, under the equivalence VdB we see that there is an atomic $\nu\mathfrak{J}$-path ν' such that $\mathbf{C}\,\mathsf{K} = \nu' C^0_{\nu\mathfrak{J}}$ lies in the region (21) and $\mathsf{K} = \Psi_{\nu'}(\mathcal{flmod}({}_{\nu'\nu}\Lambda_{\nu'\nu}))$.

Lemma 5.17. *For* K *as above, we have* $\Psi_\nu \mathsf{K} = \Psi_{\nu'\nu}(\mathit{flmod}(_{\nu'\nu}\Lambda_{\nu'\nu}))$ *and in particular the translate of* K *under the flop functor* Ψ_ν *is intermediate with respect to* $\mathit{per}(\frac{X}{Z}) \cong \mathit{flmod}\,\Lambda$.

Proof. We may assume without loss of generality that the sequence of flops ν is atomic, so by corollary 4.9 it suffices to prove that the path $\nu'\nu$ is atomic too. We show it is reduced and conclude using lemma 4.16.

Now any hyperplane crossed by ν must pass through the cone $C^0_{\mathfrak{J}} \cap \nu C^0_{\mathfrak{J}}$, and in particular the ray

$$\bigcap_{i \in \Delta \setminus \nu\mathfrak{J}} \{\varphi_\nu \alpha_i = 0\}.$$

But any two chambers in the region (21), after applying $(\varphi^\vee_\nu)^{-1}$, must lie on the same side of such a hyperplane. In particular, such a hyperplane cannot be crossed by the reduced path ν' from $\nu C^+_{\mathfrak{J}}$ to $\nu'\nu C^+_{\mathfrak{J}}$, and thus the composition $\nu'\nu$ is reduced. □

This allows us to conclude the analysis of geometric hearts, by tracking the sub-poset $[\mathit{coh}\, W, \mathit{per}(\frac{W}{Z})]_{\mathrm{alg}}$ of algebraic hearts in $[\mathit{coh}\, W, \mathit{per}(\frac{W}{Z})]$.

Proof of theorem 5.13. If $W = \nu X$ is a birational model, then for every $\mathsf{K} \in [\mathit{coh}\, W, \mathit{per}(\frac{W}{Z})]_{\mathrm{alg}}$ we have $\Psi_\nu \mathsf{K} \in \mathrm{tilt}(\mathsf{H})$ where $\mathsf{H} = \mathit{per}(\frac{X}{Z})$. It follows that the heart

$$\begin{aligned}\Psi_\nu\, \mathit{coh}\, W = \Psi_\nu \ \inf \left\{\mathsf{K} \mid \mathsf{K} \in \left[\mathit{coh}\, W, \mathit{per}(\tfrac{W}{Z})\right]_{\mathrm{alg}}\right\} \\ = \inf \left\{\Psi_\nu \mathsf{K} \mid \mathsf{K} \in \left[\mathit{coh}\, W, \mathit{per}(\tfrac{W}{Z})\right]_{\mathrm{alg}}\right\}\end{aligned}$$

lies in $\mathrm{tilt}(\mathsf{H})$ too. Just as in the $W = X$ case, examining the objects $\Psi_\nu \mathcal{O}_{C_i}(n)$ for $i \in \Delta \setminus \nu\mathfrak{J}$ and $n \in \mathbb{Z}$ shows $\mathbf{C}(\Psi_\nu\, \mathit{coh}\, W) \subseteq \nu C^0_{\mathfrak{J}}$, and realising the functionals as intersections with nef line bundles shows equality holds. Lastly, noting that $\Psi_\nu C^0_{\mathfrak{J}}$ contains $\mathsf{H}_{\mathrm{ss}}\left(\nu C^0_{\mathfrak{J}}\right) = \langle \Psi_\nu \mathcal{O}_p \mid p \in \pi^{-1}[\mathfrak{m}] \subset W\rangle$ shows that it is maximal with this heart cone, consequently $\Psi_\nu\, \overline{\mathit{coh}}\, W$ is minimal. □

The complete heart fan. An important consequence of theorem 5.13 is that we can now realise every chamber $\nu C^0_{\mathfrak{J}} \subset \Theta(\underline{\Delta}, \mathfrak{J})$ as the heart cone of a t-structure in $\mathsf{H}[-1, 0]$, and thus every maximal cone in the $\mathfrak{J}$-cone arrangement $\mathrm{Arr}(\underline{\Delta}, \mathfrak{J})$ appears in the heart fan $\mathrm{HFan}(\mathsf{H})$. Since a fan is determined by the data of its maximal cones, we have the following.

Corollary 5.18. *Under the identification* $\mathrm{Hom}(\mathbf{K}\, X, \mathbb{R}) \cong \Theta(\underline{\Delta} \setminus \mathfrak{J})$*, the heart fan of* $\mathsf{H} = \mathit{per}(\frac{X}{Z})$ *is given by the* $\mathfrak{J}$*-cone arrangement* $\mathrm{Arr}(\underline{\Delta}, \mathfrak{J})$.

Perversity arises where geometries meet (§§ 5.4 to 5.6)

§ 5.4 Perverse sheaves on partial contractions. When the flopping contraction $\pi : X \to Z$ is not irreducible (i.e. the reduced exceptional fiber has multiple components), there are intermediate partial contractions $X \to Y \to Z$ to be considered. In this subsection we establish basic structural results about perverse sheaves arising from such morphisms, following which we study semi-geometric structures in relation to the the reference heart $\mathit{per}(\frac{X}{Z}) \subset \mathbf{D}^0 X$.

Thus fix a subset $I \subset \Delta \setminus \mathfrak{J}$, equivalently a subset of reduced exceptional curves $C_I = \bigcup_{i \in I} C_i$ in X which can be contracted via the map $\tau : X \to Y$ as in (16). As noted previously, τ is a flopping contraction and in particular [Van04, theorem A] applies. Thus there is a vector bundle $\mathcal{V}(\frac{X}{Y})$ on X, furnishing a sheaf of $\mathcal{O}_Y$-algebras $\mathcal{A} = \tau_* \mathcal{E}nd\, \mathcal{V}(\frac{X}{Y})$ such that there is a derived equivalence

$$\mathrm{VdB} : \mathbf{D}^b \mathcal{A} \xrightarrow{\tau^{-1}(-) \otimes^{\mathbf{L}} \mathcal{V}(\frac{X}{Y})} \mathbf{D}^b X \tag{22}$$

where the tensor product is over $\tau^{-1}\mathscr{A}$. We write $\mathscr{P}er(\frac{X}{Y})$ for the image of the natural heart $\mathscr{C}oh\,\mathscr{A}$ under this equivalence. By [Van04, proposition 3.3.1], this category admits the description

$$\mathscr{P}er(\tfrac{X}{Y}) = \left\{ x \in \mathscr{C}oh\,X\,[0,1] \;\middle|\; \begin{array}{c} \mathbf{R}^1\tau_*(H^0x) = 0, \quad \tau_*(H^{-1}x) = 0, \\ \mathrm{Hom}(c, H^{-1}x) = 0 \text{ whenever } c \in \mathscr{C}oh\,X \text{ satisfies } \mathbf{R}\tau_*c = 0 \end{array} \right\}. \tag{23}$$

Since we are concerned with complexes supported on $\pi^{-1}[\mathfrak{m}]$, we also define the full subcategories

$$\mathbf{D}^0\mathscr{A} = \left\{ y \in \mathbf{D}^b\mathscr{A} \mid \mathrm{Supp}\, y \subseteq \varpi^{-1}[\mathfrak{m}] \right\},$$
$$coh\,\mathscr{A} = \mathscr{C}oh\,\mathscr{A} \cap \mathbf{D}^0\mathscr{A}, \qquad per(\tfrac{X}{Y}) = \mathscr{P}er(\tfrac{X}{Y}) \cap \mathbf{D}^0X.$$

The functor VdB clearly restricts to an equivalence $\mathbf{D}^0\mathscr{A} \to \mathbf{D}^0X$, thus identifying $coh\,\mathscr{A}$ with $per(\frac{X}{Y})$. The truncation functors associated to the heart $\mathscr{C}oh\,\mathscr{A} \subset \mathbf{D}^b\mathscr{A}$ evidently restrict to $\mathbf{D}^0\mathscr{A}$, showing that $coh\,\mathscr{A} \subset \mathbf{D}^0\mathscr{A}$ (and hence $per(\frac{X}{Y}) \subset \mathbf{D}^0X$) is the heart of a bounded t-structure.

Remark 5.19. The following calibration is helpful– when $I = \emptyset$ the map τ is an isomorphism and we have $\mathscr{V}(\frac{X}{Y}) = \mathscr{A} = \mathscr{O}_X$, so that $\mathscr{P}er(\frac{X}{Y}) = \mathscr{C}oh\,\mathscr{A} = \mathscr{C}oh\,X$. In this case we have $\mathbf{D}^0\mathscr{A} = \mathbf{D}^0X$ by definition. On the other extreme, when $I = \Delta \setminus \mathfrak{J}$ the map ϖ is an isomorphism and the sheaf $\mathscr{A}$ on $Y = \mathrm{Spec}\,R$ corresponds to the R-algebra Λ as in § 3. Since $(R, \mathfrak{m})$ is complete local, a complex of R-modules has support in $[\mathfrak{m}]$ if and only if its total cohomology has finite length, i.e. $\mathbf{D}^0\mathscr{A} = \mathbf{D}^{\mathrm{fl}}\Lambda$.

Characterising perversity. We now show that the 'algebraic part' of $per(\frac{X}{Y}) \subset \mathbf{D}^0X$ is generated by finitely many simple perverse sheaves which depend only on a neighbourhood of the contracted curves, and further can be read off as a subcategory of $per(\frac{X}{Z})$. For simplicity we first consider the case when $C_I \subset X$ is connected, i.e. τ is an isomorphism away from a point $p = \tau(C_I) \in Y$. Write $\underline{C}_I = \tau^{-1}(p)$ for the scheme theoretic exceptional fiber of τ, this has underlying reduced subscheme C_I. Then we have the following.

Proposition 5.20. *Let $C_I \subset X$ be a connected component of the reduced exceptional fiber in X, and $\tau : X \to Y$ the crepant contraction of C_I. For any complex of coherent sheaves $x \in \mathbf{D}^0X$ supported within C_I, the following are equivalent.*

(1) The complex x lies in $per(\frac{X}{Z})$, i.e. x is perverse with respect to the contraction $\pi : X \to Z$.

(2) The complex x lies in $per(\frac{X}{Y})$, i.e. x is perverse with respect to the contraction $\tau : X \to Y$.

(3) The complex x is filtered by the sheaves $\mathscr{O}_{C_i}(-1)$ for $i \in I$ and the complex $\omega_{\underline{C}_I}[1]$ (the suspended canonical sheaf of scheme-theoretic exceptional fiber of τ).

To prove this, we first establish the following lemma which reduces conditions involving the null-category $\{c \in \mathscr{C}oh\,X \mid \mathbf{R}\tau_*c = 0\}$ (such as those appearing in the descriptions (5), (23)) to checks on a finite collection of objects.

Lemma 5.21. *Let $\tau : X \to Y$ be the crepant contraction of C_I and $c \in \mathscr{C}oh\,X$ be such that $\mathbf{R}\tau_*c = 0$. Then c is filtered by the sheaves $\mathscr{O}_{C_i}(-1)$ for $i \in I$.*

Proof. Note if c is as given, then we have $\mathbf{R}\pi_*c = 0$ and hence $c \in \mathscr{P}er(\frac{X}{Z})$ from the description (5). Further c is supported within $\pi^{-1}[\mathfrak{m}]$. Thus across the equivalence (4), we see that c thus corresponds to some finite length Λ-module (which we again denote by c). In particular c is filtered by the simples of $H = per(\frac{X}{Z})$ and the problem is reduced to showing that the only simples which can occur in a composition series for c are the S_i for $i \in I$.

But by [Wem18, theorem 2.15], Λ can be expressed as the quotient of the path algebra of a quiver with vertex set $\underline{\Delta} \setminus \mathfrak{J}$, and the simple representation supported on vertex i is precisely the module S_i. Further writing $e_i \in \Lambda$ for the vertex idempotent at $i \in \underline{\Delta} \setminus \mathfrak{J}$, [Wem18, proposition 2.14] shows that a Λ-module x is annihilated by $\sum_{i \notin I} e_i$ if and only if the corresponding complex $x \in \mathscr{P}er(\frac{X}{Z})$ satisfies $\mathbf{R}\tau_*x = 0$. In particular, the Λ-module c is annihilated by $\sum_{i \notin I} e_i$, and hence is filtered only by the vertex simples S_i for $i \in I$ as required. □

We then have the following.

Proof of proposition 5.20, (1) $\iff$ (2). Let x be a complex of coherent sheaves with support in C_I. Since the objects of both $\mathit{per}\binom{X}{Y}$ and $\mathit{per}\binom{X}{Z}$ can be described as two-term complexes of coherent sheaves on X, we can assume x lies in $\mathit{coh}\,X[0,1]$ and for brevity write $H^{-i}_{\mathit{coh}X}(x) = x_i$ for its cohomology sheaves.

If $x \in \mathit{per}\binom{X}{Y}$, then by the description (23) we see that $\tau_*(x_1) = 0$ and hence $\pi_*(x_1) = 0$.

Likewise we have $\mathrm{Hom}(\mathcal{O}_{C_j}(-1), x_1) = 0$ whenever C_j is contracted by τ, because $\mathbf{R}\tau_*\mathcal{O}_{C_j}(-1) = 0$. On the other hand if C_j is not contracted by τ and we have a non-zero morphism $f : \mathcal{O}_{C_j}(-1) \to x_1$, then the image $\mathrm{im}\, f$ is a non-zero subsheaf of x_1 supported within the finite collection of points $C_j \cap (\bigcup_{i\in I} C_i)$. But in that case, $\pi_*(\mathrm{im}\, f)$ is a non-zero subsheaf of $\pi_*(x_1) = 0$, a contradiction. Thus in fact $\mathrm{Hom}(\mathcal{O}_{C_j}(-1), x_1) = 0$ for all exceptional curves C_j, and hence by lemma 5.21 we have $\mathrm{Hom}(c, x_1) = 0$ whenever $c \in \mathit{Coh}\,X$ satisfies $\mathbf{R}\pi_* c = 0$.

Lastly, examining the Leray spectral sequence

$$\mathbf{R}^p\varpi_* \circ \mathbf{R}^q\tau_*(x_0) \Rightarrow \mathbf{R}^{p+q}\pi_*(x_0)$$

(which degenerates since all maps have fiber dimension $\leqslant 1$) shows that $\mathbf{R}^1\pi_*(x_0)$ is filtered by $\varpi_* \circ \mathbf{R}^1\tau_*(x_0)$ ($= 0$ since $\mathbf{R}^1\tau_*(x_0)$ vanishes) and $\mathbf{R}^1\varpi_* \circ \tau_*(x_0)$ ($= 0$ since $\tau_*(x_0)$ is supported on a zero-dimensional subset of Y). Thus we have $\mathbf{R}^1\pi_*(x_0) = 0$ as well, and hence $x \in \mathit{per}\binom{X}{Z}$.

Conversely if $x \in \mathit{per}\binom{X}{Z}$, we see that $\tau_*(x_1)$ is a coherent sheaf on Y with zero-dimensional support (contained in the image of contracted curves) such that $\varpi_* \circ \tau_*(x_1) = \pi_*(x_1) = 0$. This is possible only if $\tau_*(x_1) = 0$. Likewise since $\mathbf{R}^1\pi_*(x_0) = 0$, the Leray sequence shows we have $\varpi_* \circ \mathbf{R}^1\tau_*(x_0) = 0$ and hence $\mathbf{R}^1\tau_*(x_0) = 0$. Lastly if $c \in \mathit{Coh}\,X$ is such that $\mathbf{R}\tau_* c = 0$ then we have $\mathbf{R}\pi_* c = \mathbf{R}\varpi_* \circ \mathbf{R}\tau_* c = 0$, and hence $\mathrm{Hom}(c, x_1) = 0$. This shows $x \in \mathit{per}\binom{X}{Y}$ as required. ☐

The equivalence (2) $\iff$ (3) is a consequence of [Van04, proposition 3.1.4], which states that membership for $\mathit{per}\binom{X}{Y}$ can be checked locally on Y. This allows us to reduce the problem to a formal neighbourhood $Z_I \hookrightarrow Y$ of p, i.e. the spectrum of the complete local ring $\widehat{\mathcal{O}}_{Y,p}$. The scheme Z_I has an isolated cDV singularity at its closed point, and the restriction of τ is a flopping contraction $X_I \to Z_I$ with reduced exceptional fiber $C_I \subset X_I$. We can therefore consider the full subcategory $\mathbf{D}^0 X_I \subset \mathbf{D}^b X_I$ of complexes supported on C_I.

Lemma 5.22. *The restriction (i.e. pullback) functor* $\mathbf{D}^0 X \to \mathbf{D}^0 X_I$ *gives an equivalence of the category* $\mathbf{D}^0 X_I$ *with the full subcategory of complexes in* $\mathbf{D}^0 X$ *supported within* C_I.

Proof. The schemes X and X_I have isomorphic completions along C_I, and we write $\mathfrak{X}$ for the associated Noetherian formal scheme. Now [Orl11, lemma 2.1] shows that the category $\mathbf{D}^0 X_I$ is the bounded derived category of the Abelian category

$$\mathit{coh}(X_I) = \{x \in \mathit{Coh}\,X_I \mid \mathrm{Supp}(x) \subset C_I\},$$

whilst the subcategory of $\mathbf{D}^0 X$ containing complexes supported within C_I is the bounded derived category of

$$\mathit{Coh}_{C_I}(X) = \{x \in \mathit{Coh}\,X \mid \mathrm{Supp}(x) \subset C_I\}.$$

But proposition 2.8 *ibid.* shows that both categories above are equivalent (via pullback along the canonical map associated to completion) to the category of torsion coherent sheaves on $\mathfrak{X}$. In particular, the restriction functor $\mathit{Coh}_{C_I}(X) \to \mathit{coh}(X_I)$ is an exact equivalence and passing to bounded derived categories gives the result. ☐

Thus we can identify $\mathbf{D}^0 X_I$ and its subcategories $\mathit{coh}\,X_I$, $\mathit{per}\binom{X_I}{Z_I}$ with their images in $\mathbf{D}^0 X$. In particular, $\mathit{per}\binom{X_I}{Z_I}$ (as a subcategory of $\mathbf{D}^0 X$) is an algebraic Abelian category equal to the extension-closure of its simple objects $\omega_{\underline{C}_I}[1]$, $\mathcal{O}_{C_i}(-1)$ for $i \in I$. The following is then immediate.

Proof of proposition 5.20, (2) $\iff$ (3). By [Van04, proposition 3.1.4], the membership of any complex in $\mathcal{per}\binom{X}{Y}$ can be checked locally with respect to the flat topology on Y, in particular with respect to the flat cover $Y \setminus \{p\}$, Z_I. Now if x is supported within contracted curves, the restriction of x to $Y \setminus \{p\}$ vanishes and thus x lies in $\mathcal{per}\binom{X}{Y}$ if and only if its restriction to X_I (i.e. x viewed as an object of $\mathbf{D}^0 X_I$) lies in $\mathcal{per}\left(\frac{X_I}{Z_I}\right)$, equivalently if x is filtered by the simples of $\mathcal{per}\left(\frac{X_I}{Z_I}\right)$. □

It is straightforward to generalise this to partial contractions $\tau : X \to Y$ associated to possibly disconnected collections of exceptional curves $C_I \subset X$, since any complex in $\mathbf{D}^0 X$ decomposes into a direct sum of complexes with connected support. In particular we can define the category

$$\mathcal{per}_I\binom{X}{Z} = \left\{ x \in \mathcal{per}\binom{X}{Z} \;\middle|\; \operatorname{Supp} x \subseteq \bigcup_{i \in I} C_i \right\}$$

and note that if C_I has connected components $C_{I_1}, \ldots, C_{I_k}$, then this decomposes as

$$\mathcal{per}_I\binom{X}{Z} = \bigoplus_{J \in \{I_1, \ldots, I_k\}} \mathcal{per}_J\binom{X}{Z}.$$

But following proposition 5.20, each $\mathcal{per}_J\binom{X}{Z}$ is equivalent to the category $\mathcal{per}\left(\frac{X_J}{Z_J}\right)$ associated to a formal neighbourhood Z_J of the point $p_J = \tau(C_J) \in Y$, and in particular is generated by finitely many complexes which depend only on the scheme theoretic exceptional fiber $\underline{C}_J$ overlying C_J. Further, considering the flat cover $Z_{I_1}, \ldots, Z_{I_k}, Y \setminus \{p_{I_1}, \ldots, p_{I_k}\}$ of Y shows that a complex x supported within C_I lies in $\mathcal{per}\binom{X}{Y}$ if and only if its restriction to each X_J lies in $\mathcal{per}\left(\frac{X_J}{Z_J}\right)$, and this gives us the descriptions

$$\begin{aligned}\mathcal{per}_I\binom{X}{Z} &= \left\langle \{\mathcal{O}_{C_i}(-1) \mid i \in I\} \cup \{\omega_{\underline{C}_J}[1] \mid J = I_1, \ldots, I_k\} \right\rangle \\ &= \left\{ x \in \mathcal{per}\binom{X}{Y} \;\middle|\; \operatorname{Supp} x \subseteq \bigcup_{i \in I} C_i \right\}.\end{aligned}$$

A consequence of the equivalence is that the property of a complex $x \in \mathcal{coh}\, X[0,1]$ being 'perverse' can be checked with respect to any partial contraction which maps $\operatorname{Supp}(x)$ to a (finite collection of) points. That is to say, if $\tau : X \to Y$ and $\tau' : X \to Y'$ are two crepant partial contractions which contract all the curves containing $\operatorname{Supp}(x)$, then we have $x \in \mathcal{per}\binom{X}{Y}$ if and only if $x \in \mathcal{per}\binom{X}{Y'}$.

Since $\mathcal{per}\binom{X}{Y}$ away from the contracted curves should mimic $\mathcal{per}\binom{X}{X} = \mathcal{coh}\, X$, we then have the following result which reduces the construction of $\mathcal{per}\binom{X}{Y}$ to a binary choice between $\mathcal{coh}\, X$ and $\mathcal{per}\binom{X}{Z}$ on each C_i, effectively eliminating the need to consider the geometry of the non-commutative scheme $(Y, \mathcal{A})$.

Theorem 5.23. *If $\tau : X \to Y$ is the crepant contraction of the exceptional subset $C_I \subset X$, then the associated heart $\mathcal{per}\binom{X}{Y}$ is the smallest extension-closed subcategory of $\mathbf{D}^0 X$ containing the full subcategory $\mathcal{per}_I\binom{X}{Z}$ and all coherent sheaves on X that are supported within the uncontracted curves $\bigcup_{i \notin I} C_i$. In other words,*

$$\mathcal{per}\binom{X}{Y} = \left\langle \underbrace{\left\{ x \in \mathcal{per}\binom{X}{Z} \;\middle|\; \operatorname{Supp} x \subseteq \bigcup_{i \in I} C_i \right\}}_{\mathcal{per}_I\binom{X}{Z}} \cup \left\{ x \in \mathcal{coh}\, X \;\middle|\; \operatorname{Supp} x \subseteq \bigcup_{i \notin I} C_i \right\} \right\rangle.$$

Proof. The category $\mathcal{per}_I\binom{X}{Z}$ clearly lies in $\mathcal{per}\binom{X}{Y}$, while if $x \in \mathcal{coh}\, X$ is supported within $\bigcup_{i \notin I} C_i$ then $\mathbf{R}\tau_* x$ is a sheaf on Y and hence $x \in \mathcal{per}\binom{X}{Y}$. To furnish the required description of $\mathcal{per}\binom{X}{Y}$, it thus suffices to show that every object in the category is an extension of complexes of the two given forms.

Now an arbitrary complex $x \in \mathcal{per}\binom{X}{Y}$ can be written as an extension of its cohomology objects with respect to $\mathcal{coh}\, X$, namely $x_0 = H^0(x)$ and $x_1[1] = H^{-1}(x)[1]$. It is clear from (23) that x_0, $x_1[1]$ are themselves contained in $\mathcal{per}\binom{X}{Y}$. Further we have $\tau_*(x_1) = 0$ and hence x_1 must be supported within the contracted curve $\bigcup_{i \in I} C_i$, from which it follows that $x_1[1] \in \mathcal{per}_I\binom{X}{Z}$.

On the other hand, writing $\mathcal{I} \subset \mathcal{O}_X$ for the ideal sheaf of the closed subscheme $\bigcup_{i \in I} C_i$, note that $y = \mathcal{I}^n \cdot x_0$ (i.e. the image of the natural map $\mathcal{I}^n \otimes x_0 \to x_0$) is a subsheaf of x_0 (in particular, an object of $\mathcal{coh}\, X$) supported within $\bigcup_{i \notin I} C_i$ for $n \gg 0$. The quotient x_0/y is clearly supported within $\bigcup_{i \in I} C_i$. Further, applying τ_* to this quotient and examining the long exact sequence shows $\mathbf{R}^1\tau_*(x_0/y)$ vanishes since $\mathbf{R}^1\tau_*(x_0)$ does, i.e. x_0/y lies in $\mathcal{per}(\frac{X}{Y})$ (and hence in $\mathcal{per}_I(\frac{X}{Z})$). This concludes our proof that x is an extension of objects of the required form. □

Simple objects and K-theory. It is immediate from theorem 5.23 that an object of $\mathcal{per}(\frac{X}{Y})$ is simple if and only if it is simple in some summand $\mathcal{per}_J(\frac{X}{Z}) \subset \mathcal{per}_I(\frac{X}{Z})$ or a simple coherent sheaf supported on the uncontracted locus, i.e. we have the following.

Corollary 5.24. *For $\tau : X \to Y$ the crepant contraction of $C_I \subset X$, the simple objects of $\mathcal{per}(\frac{X}{Y})$ are precisely*

skyscrapers $\mathcal{O}_p$ at closed points $p \notin \bigcup_{i \in I} C_i$,

the sheaves $\mathcal{O}_{C_j}(-1)$ for $j \in I$, and

suspended canonical sheaves $\omega_{\underline{C}_J}[1]$ for each scheme-theoretic exceptional fiber $\underline{C}_J$ of τ.

For the rest of this subsection, we make the additional assumption $I \cup \nu\mathfrak{J} \neq \Delta$, i.e. that the partial contraction $\tau : X \to Y$ associated to $C_I \subset X$ does not contract every curve. We continue to write $C_{I_1}, \dots, C_{I_k}$ for the connected components of C_I, and $\underline{C}_J$ for the scheme theoretic fiber overlying a connected component C_J.

Identifying $\mathbf{K}\, X$ with $\mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$, we can then consider the heart cone of $K = \mathcal{per}(\frac{X}{Y}) \in$ t-str$(\mathbf{D}^0 X)$.

Lemma 5.25. *Under the given assumptions, the heart $K = \mathcal{per}(\frac{X}{Y})$ has heart cone $\mathbf{C}\, K = C^0_{\mathfrak{J} \cup I}$.*

Proof. If θ lies in the heart cone of $\mathcal{per}(\frac{X}{Y})$, then considering the sheaves $\mathcal{O}_{C_i}(n)$ for $n \in \mathbb{Z}$ on uncontracted curves and the sheaves $\mathcal{O}_{C_i}(-1)$ on contracted curves shows θ lies in the chamber $C^0_{\mathfrak{J}}$. In particular, θ necessarily vanishes on the class $\delta_{\mathfrak{J}}$ of skyscraper sheaves. But for a connected component $C_J \subset C_I$ and a closed point $p \in C_J$, we see from lemma 5.6 that $\mathcal{O}_p$ admits a filtration by the simples of $\mathcal{per}_J(\frac{X}{Z})$ in which every simple appears at least once. Thus in $\mathbf{K}$-theory we have

$$[\mathcal{O}_p] = \left[\omega_{\underline{C}_J}[1]\right] + \sum_{i \in J} m_i\, [\mathcal{O}_{C_i}(-1)] \quad = \left[\omega_{\underline{C}_J}[1]\right] + \sum_{i \in J} m_i\, \alpha_i \tag{24}$$

for positive integers m_i. Since θ is non-negative on each class appearing in the above expression and in particular vanishes on $[\mathcal{O}_p]$, it must vanish on every class on the right-hand side. Repeating the argument on all connected components of C_I shows $\theta(\alpha_i) = 0$ whenever $i \in I$, and thus the heart cone $\mathbf{C}\, K$ is contained in $C^0_{\mathfrak{J} \cup I}$.

Conversely if θ lies in $C^0_{\mathfrak{J} \cup I}$, then in particular θ lies in the heart cone of $\mathcal{coh}\, X$ and therefore $\theta[x] \geqslant 0$ whenever x is a coherent sheaf supported on uncontracted curves in X. On the other hand if $x \in \mathcal{per}(\frac{X}{Y})$ is supported in C_I, then we may assume x is supported in some connected component $C_J \subset C_I$ so that x lies in $\mathcal{per}_J(\frac{X}{Z})$. But then proposition 5.20 and the expression (24) show that the $\mathbf{K}$-theory class $[x]$ can be expressed as some integral linear combination

$$[x] = n_\delta\, [\mathcal{O}_p] + \sum_{i \in J} n_i\, [\mathcal{O}_{C_i}(-1)] \quad = n_\delta\, \delta_{\mathfrak{J}} + \sum_{i \in J} n_i\, \alpha_i,$$

so that $\theta[x] = 0$. Since an arbitrary complex $x \in \mathcal{per}(\frac{X}{Y})$ is an extension of objects of the above forms (theorem 5.23), it follows that $\theta[x] \geqslant 0$ and thus $C^0_{\mathfrak{J} \cup I}$ is equal to the heart cone of $\mathcal{per}(\frac{X}{Y})$. □

Note in particular that for a generic $\theta \in C^0_{\mathfrak{J} \cup I}$, every simple of $K = \mathcal{per}(\frac{X}{Y})$ is orthogonal to θ in $\mathbf{K}$-theory and is therefore θ-stable. We now show that an object is θ-semistable if and only if it is filtered by the simples– note this is not immediate as K is not algebraic when some curve is left uncontracted.

Lemma 5.26. *For* $\mathsf{K} = \mathcal{per}\binom{X}{Y}$ *as given and a generic vector* $\theta \in C^{0}_{\mathfrak{J}\cup I}$*, an object of* K *is* θ*-semistable if and only if it is filtered by the simple objects of* K*. In other words, the category of* θ*-semistables in* K *is given by*

$$\mathsf{K}_{ss}(\theta) = \bigoplus_{J=I_1,\dots,I_k} \mathcal{per}_J\binom{X}{Z} \;\oplus\; \bigoplus_{p\in C\setminus C_I} \langle \mathcal{O}_p \rangle.$$

Proof. Note that θ lies in the heart cone of K, thus an object $k \in \mathsf{K}$ is θ-semistable if and only if $\theta[k] = 0$. Then one implication is clear since the vector θ vanishes on every simple object of $\mathcal{per}\binom{X}{Y}$.

To show the converse, note that theorem 5.23 shows that any $x \in \mathcal{per}\binom{X}{Z}$ is an extension of the simples of $\mathcal{per}_I\binom{X}{Z}$ (which are also simples of $\mathcal{per}\binom{X}{Y}$) by some coherent sheaf supported on the uncontracted curves in X. Thus we show that if $x \in \mathcal{coh}\,X$ is supported within $\bigcup_{i\notin I} C_i$ and satisfies $\theta[x] = 0$, then x is an extension of skyscraper sheaves i.e. has zero-dimensional support.

Indeed if not, then some curve C_j $(j \notin I)$ lies in $\mathrm{Supp}(x)$ and thus there is an exact sequence of coherent sheaves $0 \to x' \to x \to \mathcal{O}_{C_j}(n) \to 0$ for $n \ll 0$. Clearly, all three sheaves in this sequence are supported within $\bigcup_{i\notin I} C_i$ and thus this sequence is also exact in $\mathcal{per}\binom{X}{Y}$. But then since x is θ-semistable, we have $\theta\left[\mathcal{O}_{C_j}(-1)\right] = \theta(\alpha_j) \leqslant 0$ which contradicts the fact that θ is generic in $C^{0}_{\mathfrak{J}\cup I}$. ☐

§ 5.5 Many flavours of semi-geometric hearts. Consider a partial contraction $\tau : X \to Y$ which contracts the collection of exceptional curves $C_I \subset X$ with connected components $C_I = C_{I_1}, \dots, C_{I_k}$. The associated category of perverse sheaves $\mathsf{K} = \mathcal{per}\binom{X}{Y}$, being an amalgamation of $\mathcal{per}\binom{X}{Z}$ and $\mathcal{coh}\,X$, is Noetherian and thus the extension-closure of its simple objects (i.e. the category $\mathsf{K}_{ss}\left(C^{0}_{\mathfrak{J}\cup I}\right)$) is a torsion class in K. Taking the corresponding tilt of $\mathcal{per}\binom{X}{Y}$ defines the *reversed semi-geometric heart*

$$\overline{\mathcal{per}}\binom{X}{Y} = \left\langle \mathcal{per}_I\binom{X}{Z}[-1] \;\cup\; \left\{ x \in \overline{\mathcal{coh}}\,X \;\middle|\; \mathrm{Supp}\,x \subseteq \bigcup_{i\notin I} C_i \right\} \right\rangle,$$

which is an Artinian t-structure on $\mathbf{D}^0X$.

Considering t-structures in the interval $\left[\overline{\mathcal{per}}\binom{X}{Y}, \mathcal{per}\binom{X}{Y}\right] \subset \text{t-str}(\mathbf{D}^0X)$ then opens the floodgates to a plethora of semi-geometric hearts.

Theorem 5.27. *Given* $\mathsf{K} \in \left[\overline{\mathcal{per}}\binom{X}{Y}, \mathcal{per}\binom{X}{Y}\right]$*, each restriction* $\mathsf{K} \cap \mathbf{D}^0X_J$ *(*$J = I_1, \dots, I_k$*) is a tilt of* $\mathcal{per}_J\binom{X}{Z}$*.*

Further, K *is uniquely determined by these restrictions and the set* $\{p \in \pi^{-1}[\mathfrak{m}] \setminus C_I \mid \mathcal{O}_p[-1] \in \mathsf{K}\}$*, and this determines an order- and brick-label–preserving bijection of posets*

$$\left[\overline{\mathcal{per}}\binom{X}{Y}, \mathcal{per}\binom{X}{Y}\right] \quad\longrightarrow\quad \prod_{J=I_1,\dots,I_k} \mathrm{tilt}\left(\mathcal{per}_J\binom{X}{Z}\right) \;\times\; \mathrm{Bool}\left(\pi^{-1}[\mathfrak{m}] \setminus C_I\right).$$

Here Bool(−) denotes the boolean lattice on closed points, and the poset on the right hand side is given the product order i.e. $(a_i) \leqslant (b_i)$ if and only if $a_i \leqslant b_i$ for every i.

Proof. By lemma 2.13 and the discussion following it, hearts $\mathsf{K} \in \left[\overline{\mathcal{per}}\binom{X}{Y}, \mathcal{per}\binom{X}{Y}\right]$ are in bijection with torsion classes $\mathsf{T} \subset \mathcal{per}\binom{X}{Y}$ that satisfy $\mathsf{T} \subseteq \mathsf{K}_{ss}\left(C^{0}_{\mathfrak{J}\cup I}\right)$. For such a T, lemma 5.26 induces the decomposition

$$\mathsf{T} = \bigoplus_{J=I_1,\dots,I_k} \mathsf{T} \cap \mathcal{per}_J\binom{X}{Z} \;\oplus\; \bigoplus_{p\in C\setminus C_I} \mathsf{T} \cap \langle \mathcal{O}_p \rangle,$$

and each summand in the above decomposition is closed under extensions and factors (in $\mathsf{K}_{ss}\left(C^{0}_{\mathfrak{J}\cup I}\right)$) since T is so. It follows that each $\mathsf{T}_J = \mathsf{T} \cap \mathcal{per}_J\binom{X}{Z}$ is a torsion class in $\mathcal{per}_J\binom{X}{Z}$, and $\mathsf{K} \cap \mathbf{D}^0X_J$ is the tilt of $\mathcal{per}_J\binom{X}{Z} = \mathcal{per}\binom{X}{Y} \cap \mathbf{D}^0X_J$ in $\mathsf{T}_J = \mathsf{T} \cap \mathbf{D}^0X_J$. Further T is clearly determined the summands, i.e. the desired bijection follows. ☐

Thus we have many 'flavours' of semi-geometric hearts, each arising by mutating the algebraic part of $\mathcal{per}\left(\frac{X}{Y}\right)$ and possibly tilting at skyscraper sheaves in its geometric locus. Further we can repeat these constructions on other birational models and translate the resulting t-structures across the flop functor, this gives more hearts in $\mathbf{D}^0X$ of a semi-geometric nature. In what follows we analyse these in relation to the reference heart $\mathcal{per}\left(\frac{X}{Z}\right)$ showing that some of these are intermediate and manifest as non-maximal cones in the heart fan.

§5.6 Intermediacy of semi-geometric hearts. Throughout this subsection, we consider a birational model $W = \nu X$ of X and the crepant contraction $W \to Y$ of a subset of exceptional curves $C_I = \bigcup_{i\in I} C_i \subset W$ determined by the Dynkin data $I \subset \Delta \setminus \nu\mathfrak{J}$. Thus we can consider semi-geometric hearts in $\mathbf{D}^0W$, and transport them across the flop functor Ψ_ν to obtain an interval of t-structures

$$\Psi_\nu\left[\overline{\mathcal{per}}\left(\tfrac{W}{Y}\right), \mathcal{per}\left(\tfrac{W}{Y}\right)\right] = \{\Psi_\nu K \mid \overline{\mathcal{per}}\left(\tfrac{W}{Y}\right) \leqslant K \leqslant \mathcal{per}\left(\tfrac{W}{Y}\right)\} \quad \subset \text{t-str}(\mathbf{D}^0X).$$

By theorem 5.23, the 'classic' semi-geometric heart $\Psi_\nu\, \mathcal{per}\left(\frac{W}{Y}\right)$ lies in the interval $\Psi_\nu\left[\mathcal{coh}\, W, \mathcal{per}\left(\frac{W}{Z}\right)\right]$ and is therefore intermediate with respect to $H = \mathcal{per}\left(\frac{X}{Z}\right)$ since both $\Psi_\nu\, \mathcal{coh}\, W$ and $\Psi_\nu\, \mathcal{per}\left(\frac{W}{Z}\right)$ are so. Further, if we make the additional assumption $I \cup \nu\mathfrak{J} \neq \Delta$ (i.e. not every curve in W is contracted) then combining lemma 2.10 with lemma 5.25 immediately shows $K = \Psi_\nu\, \mathcal{per}\left(\frac{W}{Z}\right)$ has heart cone

$$\mathbf{C}\, K = (\varphi_\nu^\vee)^{-1} C^0_{\nu\mathfrak{J}\cup I} \quad = \nu C^0_{\mathfrak{J}} \cap \bigcap_{i\in I}\{\varphi_\nu \alpha_i = 0\}.$$

In particular, every non-zero and non-maximal cone in $\mathrm{Arr}^0(\underline{\Delta}, \mathfrak{J}) \subset \mathrm{HFan}(H)$ is realised as the heart cone of an intermediate semi-geometric heart.

The reversed semi-geometric heart, however, often lies outside the range $H[-1, 0]$. Indeed in the extreme case $I = \Delta \setminus \nu\mathfrak{J}$, we have $\Psi_\nu \overline{\mathcal{per}}\left(\frac{W}{Y}\right) = \Psi_\nu\, \mathcal{per}\left(\frac{W}{Z}\right)[-1]$ and this is intermediate with respect to H if and only if ν is trivial (i.e. $W = X$).

More generally, the characterisation of when $\Psi_\nu \overline{\mathcal{per}}\left(\frac{W}{Y}\right)$ lies in $H[-1, 0]$ reduces to a combinatorial condition on the path ν.

Theorem 5.28. *Consider a birational model* $W = \nu X$ *of* X *and let* $\tau : W \to Y$ *be the crepant contraction of a collection of exceptional curves* $C_I \subset W$*. Writing* $\sigma \subset \Theta(\underline{\Delta} \setminus \mathfrak{J})$ *for the heart cone of* $\Psi_\nu\, \mathcal{per}\left(\frac{W}{Y}\right)$*, the following statements are equivalent.*

(1) The shortest $\mathfrak{J}$*-path* μ *satisfying* $\sigma \subset \mu C^0_{\mathfrak{J}}$ *gives a sequence of flops from* X *to* W*, i.e.* $W = \mu X$*.*

(2a) The heart $\Psi_\nu\, \mathcal{per}\left(\frac{W}{Y}\right)$ *is the supremum of the collection of geometric hearts* $\left\{\Psi_\upsilon\, \mathcal{coh}(\upsilon X) \mid \sigma \subset \upsilon C^0_{\mathfrak{J}}\right\}$*.*

(2b) The heart $\Psi_\nu\, \mathcal{per}\left(\frac{W}{Y}\right)$ *is equal to the numerically defined tilt* $H^{tt}(\sigma)$*.*

(3) The reversed semi-geometric heart $\Psi_\nu \overline{\mathcal{per}}\left(\frac{W}{Y}\right)$ *is intermediate with respect to* $H = \mathcal{per}\left(\frac{X}{Z}\right)$*.*

(3a) The heart $\Psi_\nu \overline{\mathcal{per}}\left(\frac{W}{Y}\right)$ *is the infimum of the collection of geometric hearts* $\left\{\Psi_\upsilon\, \overline{\mathcal{coh}}(\upsilon X) \mid \sigma \subset \upsilon C^0_{\mathfrak{J}}\right\}$*.*

(3b) The heart $\Psi_\nu \overline{\mathcal{per}}\left(\frac{W}{Y}\right)$ *is equal to the numerically defined tilt* $H_{tt}(\sigma)$*.*

(4) For generic $\theta \in \sigma$*, an object of* H *is* θ*-stable if and only if it is a simple of* $\Psi_\nu\, \mathcal{per}\left(\frac{W}{Y}\right)$*.*

Further for any non-zero cone $\sigma \in \mathrm{Arr}^0(\underline{\Delta}, \mathfrak{J})$*, there is a unique birational model* $W = \nu X$ *and a unique partial contraction* $W \to Y$ *for which* $\sigma = \mathbf{C}(\Psi_\nu\, \mathcal{per}\left(\frac{W}{Y}\right))$ *and the above statements hold.*

Before proving the various equivalences we sketch a proof of the final claim i.e. we show that for any $\sigma \in \mathrm{Arr}(\Delta, \mathfrak{J})$ there is a unique birational model and partial contraction satisfying the said conditions. Now each chamber $\nu C^0_{\mathfrak{J}}$ containing σ as a face uniquely determines a subset $I \subset \Delta \setminus \nu\mathfrak{J}$ such that $\varphi_\nu^\vee \sigma = C^0_{\nu\mathfrak{J}\cup I}$, and the data (ν, I) together determine a partial contraction $\nu X \to \nu X_{\mathrm{con}}$ such that $\mathbf{C}\left(\Psi_\nu\, \mathcal{per}\left(\frac{\nu X}{\nu X_{\mathrm{con}}}\right)\right) = \sigma$.

One checks this assignment is well-defined (i.e. choosing a different path ν' from $C^0_{\mathfrak{J}}$ to $\nu C^0_{\mathfrak{J}}$ yields the same partial contraction), thus obtaining a bijective correspondence

$$\left\{\text{Chambers } \nu C^0_{\mathfrak{J}} \text{ containing } \sigma \text{ as a face}\right\} \Longleftrightarrow \left\{\text{Partial contractions } \nu X \to \nu X_{\mathrm{con}} \text{ with } \sigma = \mathbf{C}\left(\Psi_\nu\, \mathcal{per}\left(\tfrac{\nu X}{\nu X_{\mathrm{con}}}\right)\right)\right\}$$

where the inverse map assigns a partial contraction $\nu X \to \nu X_{\mathrm{con}}$ to the heart cone $\mathbf{C}(\Psi_\nu\, \mathcal{coh}\, \nu X) = \nu C^0_{\mathfrak{J}}$.

Now there is a unique chamber $\tau \in \mathrm{Arr}^0(\underline{\Delta}, \mathfrak{J})$ such that σ is a face of τ and the chambers $\tau, C^0_{\mathfrak{J}}$ lie on the same side of any root hyperplane passing through σ. Considering the analogue of lemma 4.16 for the induced intersection arrangement $\mathrm{Arr}^0(\underline{\Delta}, \mathfrak{J}) \cong \mathrm{Arr}(\Delta, \mathfrak{J})$, one shows that a spherical $\mathfrak{J}$-path μ satisfying $\sigma \subset \mu C^0_{\mathfrak{J}}$ has minimal length among all such paths if and only if μ is an atomic path and $\tau = \mu C^0_{\mathfrak{J}}$. It follows that for such a path μ, the partial contraction $\mu X \to \mu X_{\mathrm{con}}$ is the unique one for which the statement (1) holds.

Returning to the setup of theorem 5.28, we now establish the equivalence of statements (1)–(4). The following equivalences arise from the interplay between numerical tilts and semistable categories.

Proof of (2b) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (3b). Consider the heart $K = \Psi_\nu\, \mathcal{per}(\frac{W}{Y})$ and a generic parameter $\theta \in \sigma$. To begin, note that lemma 5.26 shows that an object in K is θ-semistable if and ony if it is filtered by the simple objects of K, i.e. the statement (4) is equivalent to showing $H_{ss}\theta = K_{ss}\theta$.

But we know K is intermediate with respect to H and has heart cone σ, so corollary 2.14 shows that we have $H_{ss}\theta = K_{ss}\theta$ if and only if K is equal to $H^{tt}\theta$, thus showing (2b) $\Longleftrightarrow$ (4).

If (2b) (and hence also (4)) is true, then it is immediate that (3b) holds, since $\Psi_\nu\, \overline{\mathcal{per}}(\frac{W}{Y})$ (i.e. the tilt of K in $K_{ss}\theta$) must coincide with $H_{tt}\theta$ (i.e. the tilt of $H^{tt}\theta$ in $H_{ss}\theta$). Conversely suppose $\Psi_\nu\, \overline{\mathcal{per}}(\frac{W}{Y}) = H^{tt}\theta$. Since K lies in $[H_{tt}\theta, H^{tt}\theta]$, we see that $K \cap H_{tt}\theta[1]$ is a torsion-free class in $H_{ss}\theta$ and in particular is contained in H. But we have $K \cap H_{tt}\theta[1] = \Psi_\nu\, \mathcal{per}(\frac{W}{Y}) \cap \Psi_\nu\, \overline{\mathcal{per}}(\frac{W}{Y})[1]) = K_{ss}\theta$, so in particular $K_{ss}\theta \subset H$ and hence corollary 2.14 shows that both (2b) and (4) hold. □

Likewise, the equivalences (2a) $\Longleftrightarrow$ (2b) and (3a) $\Longleftrightarrow$ (3b) only rely on the geometry of heart cones. We show how to prove the former, the latter being similar.

Proof of (2a) $\Longleftrightarrow$ (2b). It suffices to show that $H^{tt}(\sigma)$ is the supremum of the given collection of geometric hearts. If σ is a face of $\upsilon C^0_{\mathfrak{J}} = \mathbf{C}(\Psi_\upsilon\, \mathcal{coh}(\upsilon X))$, then the inequality $\Psi_\upsilon\, \mathcal{coh}(\upsilon X) \leqslant H^{tt}(\sigma)$ clearly holds by lemma 2.12.

Conversely, choose generic vectors $\theta_\upsilon \in \upsilon C^0_{\mathfrak{J}}$ for each $\upsilon C^0_{\mathfrak{J}}$ containing σ as a face. Since the fan $\mathrm{Arr}^0(\underline{\Delta}, \mathfrak{J})$ is induced from a simplicial hyperplane arrangement, there is a tuple of positive reals (λ_υ) such that the weighted average $\theta = \sum \lambda_\upsilon \theta_\upsilon$ sits generically in the face σ. By the construction of numerical torsion theories, we see that if h lies in each torsion class $H_{tr}(\theta_\upsilon) = \Psi_\upsilon\, \mathcal{coh}(\upsilon X)[1] \cap H$, then we also have $h \in H_{tr}(\theta) = H^{tt}(\sigma)[1] \cap H$. It follows that we have $H^{tt}(\sigma) \leqslant \sup \{\Psi_\upsilon\, \mathcal{coh}(\upsilon X) \mid \sigma \subset \upsilon C^0_{\mathfrak{J}}\}$, and hence equality holds. □

Unsurprisingly, relating these convex-geometric statements to (1) leverages the control we have over compositions of atomic mutations.

Proof of (1)$\Rightarrow$(2a). First consider the case when the shortest such path μ is given by the empty word $\emptyset$, i.e. $W = X$ and $\sigma = C^0_{\mathfrak{J} \cup I}$ is a face of $C^0_{\mathfrak{J}}$. Then for $K = \mathcal{per}(\frac{X}{Y})$ and $\theta \in \sigma$ generic, the category $K_{ss}\theta$ (as computed in lemma 5.26) clearly lies in H and thus corollary 2.14 shows we have $K = H^{tt}\theta$, i.e. statement (2b) (and hence also (2a)) holds.

Now suppose W, Y, σ are as in the general situation and μ is the shortest spherical $\mathfrak{J}$-path satisfying $\sigma \subset \mu C^0_{\mathfrak{J}}$, i.e. $\varphi^\vee_\nu \sigma = C_{\mu\mathfrak{J} \cup I}$ for some $I \subset \mu\mathfrak{J}$. If (1) holds, then we have $W = \mu X$ and thus $W \to Y$ is the crepant contraction of $C_I \subset W$. The reasoning above then shows

$$\mathcal{per}(\tfrac{W}{Y}) = \sup\ \left\{\Psi_\upsilon\, \mathcal{coh}(\upsilon W) \mid C^0_{\mu\mathfrak{J}\cup I} \subset \upsilon C^0_{\mu\mathfrak{J}}\right\} \quad \in \text{t-str}(\mathbf{D}^0 W),$$

and it suffices to restrict the above expression to spherical $\mu\mathfrak{J}$-path υ that are atomic. But if υ is atomic then so is $\upsilon\mu$, since every hyperplane crossed by υ must contain σ while the choice of μ ensures that no hyperplane crossed by it contains σ. Thus have

$$\begin{aligned}\Psi_\mu \,\mathit{per}\left(\tfrac{W}{Y}\right) &= \sup\ \left\{\Psi_\mu \circ \Psi_\upsilon\, \mathit{coh}(\upsilon W) \mid C^0_{\mu\mathfrak{J}\cup I} \subset \upsilon C^0_{\mu\mathfrak{J}}\right\}\\ &= \sup\ \left\{\Psi_{\upsilon\mu}\, \mathit{coh}(\upsilon\mu X) \mid \sigma \subset \upsilon\mu C^0_{\mathfrak{J}}\right\}\end{aligned}$$

To see this is equivalent to the required statement (2b), observe that for ny spherical $\mu\mathfrak{J}$-path υ we have $C^0_{\mu\mathfrak{J}\cup I} \subset \upsilon C^0_{\mu\mathfrak{J}}$ if and only if $\sigma \subset \upsilon\mu C^0_{\mathfrak{J}}$, and every maximal chamber containing σ as a face can be realised in this way since $\varphi^\vee_\mu : \mathrm{Arr}(\underline{\Delta}, \mathfrak{J}) \to \mathrm{Arr}(\underline{\Delta}, \mu\mathfrak{J})$ is an isomorphism of fans. □

We thus have the implications (1)⇒ (2a) $\iff$ (2b) $\iff$ (4) $\iff$ (3a) $\iff$ (3b). Note that the implication (3b)⇒(3) is immediate from definitions, so we conclude the proof of theorem 5.28 as follows.

Proof of (3)⇒(4). Suppose (3) holds i.e. we have $\Psi_\nu \,\overline{\mathit{per}}\left(\tfrac{W}{Y}\right) \subset H[-1,0]$. Since $K = \Psi_\nu\, \mathit{per}\left(\tfrac{W}{Y}\right)$ also lies in $H[-1,0]$, we see that for generic $\theta \in \sigma$ the category $K_{ss}\theta = \Psi_\nu\, \mathit{per}\left(\tfrac{W}{Y}\right) \cap \Psi_\nu\, \overline{\mathit{per}}\left(\tfrac{W}{Y}\right)[1]$ lies in H, and hence by corollary 2.14 and corollary 5.24 we conclude that every object in $H_{ss}\theta$ is filtered by the simples of K i.e. (4) holds. □

Proof of (4)⇒(1). The path μ as defined in (1) gives an atomic sequence of flops from X, defining a birational model $W' = \mu X$. Further since σ is a face of $\mu C^0_{\mathfrak{J}}$, this birational model admits a partial contraction $W' \to Y'$ such that $\mathbf{C}\left(\Psi_\mu \,\mathit{per}\left(\tfrac{W'}{Y'}\right)\right) = \sigma$. The statement (1) (and hence (4)) holds for this partial contraction by construction, so we see that an object of H is θ-stable if and only if it is a simple object of $\Psi_\mu\, \mathit{per}\left(\tfrac{W'}{Y'}\right)$.

Choosing an atomic sequence of flops υ from W' to W, we note that the path υ from $\mu C^0_{\mathfrak{J}}$ to $\upsilon\mu C^0_{\mathfrak{J}} = \nu C^0_{\mathfrak{J}}$ only crosses hyperplanes containing σ while the path μ by definition never passes through a chamber containing σ. Thus the composite path $\upsilon\mu$ is also atomic and corollary 4.9 allows us to write $\Psi_\nu = \Psi_\mu \circ \Psi_\upsilon$.

Suppose (4) holds for $W \to Y$, i.e. an object of H is θ-stable if and only if it is a simple of $\Psi_\nu\, \mathit{per}\left(\tfrac{W}{Y}\right)$. Thus we have the equality of sets

$$\left\{\Psi_\mu^{-1} h \mid h \in H \text{ is } \theta\text{-stable}\right\} \quad = \left\{s \mid s \in \mathit{per}\left(\tfrac{W'}{Y'}\right) \text{ is simple}\right\} \quad = \left\{\Psi_\upsilon t \mid t \in \mathit{per}\left(\tfrac{W}{Y}\right) \text{ is simple}\right\}.$$

We need to show $W = W'$, or equivalently that υ is the empty path. If not, it has length $\geqslant 1$ so can be written as $\upsilon = \nu_i \cdot \upsilon'$ for some $\mu\mathfrak{J}$-path υ' and $i \in \Delta \setminus \upsilon'\mu\mathfrak{J}$. In other words, the final curve flopped by υ is $C_i \subset \upsilon' W'$, which has proper transform $C_{\iota(i)} \subset W$.

Further the wall $\upsilon' C^0_{\mathfrak{J}} \cap \nu_i \upsilon' C^0_{\mathfrak{J}}$ contains the cone σ, so we must have $\iota(i) \in I$. Thus the curve $C_{\iota(i)} \subset W$ is contracted by $\tau : W \to Y$.

In particular the object $t = \mathcal{O}_{C_{\iota(i)}}(-1)$ is simple in $\mathit{per}\left(\tfrac{W}{Y}\right)$, and tracking it across the flop functors using lemma 4.3 we see that

$$\begin{aligned}\Psi_\upsilon(t) &= \Psi_{\upsilon'} \circ \Psi_i(\mathcal{O}_{C_{\iota(i)}}(-1))\\ &= \Psi_{\upsilon'}\left(\mathcal{O}_{C_i}(-1)\right)[-1]\\ &\in \mathit{per}\left(\tfrac{W'}{Z}\right)[-2,-1].\end{aligned}$$

But all simples of $\mathit{per}\left(\tfrac{W'}{Y'}\right)$ must lie in $\mathit{per}\left(\tfrac{W'}{Z}\right)$, this furnishes the desired contradiction. □

In summary, we have the following analysis of non-trivial cones in $\mathrm{Arr}^0(\underline{\Delta}, \mathfrak{J})$.

Corollary 5.29. *Suppose $\sigma \in \mathrm{Arr}^0(\underline{\Delta}, \mathfrak{J})$ is a non-zero cone. Then there is a unique birational model $W = \nu X$ that sits in a partial contraction $\tau : W \to Y$ such that $\mathbf{C}\left(\Psi_\nu\, \mathit{per}\left(\tfrac{W}{Y}\right)\right) = \sigma$ and the translate of any t-structure in $[\overline{\mathit{per}}\left(\tfrac{W}{Y}\right), \mathit{per}\left(\tfrac{W}{Y}\right)]$ across the flop functor Ψ_ν is intermediate with respect to $H = \mathit{per}\left(\tfrac{X}{Z}\right)$. In this case the t-structures $\Psi_\nu\, \overline{\mathit{per}}\left(\tfrac{W}{Y}\right)$ and $\Psi_\nu\, \mathit{per}\left(\tfrac{W}{Y}\right)$ are determined numerically as*

$$\Psi_\nu\, \overline{\mathit{per}} W = H_{tt}(\sigma), \qquad \Psi_\nu\, \mathit{per} W = H^{tt}(\sigma),$$

and thus an intermediate heart $\mathsf{K} \in \operatorname{tilt}(\mathsf{H})$ *satisfies* $\sigma \subset \mathbf{C}\,\mathsf{K}$ *if and only if it lies in the interval* $\Psi_\nu\left[\overline{\mathit{per}}\left(\frac{W}{Y}\right), \mathit{per}\left(\frac{W}{Y}\right)\right]$.

The brick-labelled partial order on $\Psi_\nu\left[\overline{\mathit{per}}\left(\frac{W}{Y}\right), \mathit{per}\left(\frac{W}{Y}\right)\right]$ can then be described using theorem 5.27. Combined with corollaries 4.7 and 5.14, for any non-zero $\theta \in \Theta(\underline{\Delta}, \mathfrak{J})$ we thus have a complete description of the interval $[\mathsf{H}_{tt}\theta, \mathsf{H}^{tt}\theta] \subset \operatorname{tilt}(\mathsf{H})$ and the brick labels for covering relations in it.

§ 6 Dynamics of the nef monoid

When studying a heart in relation to the poset of t-structures, one can equivalently study more amiable families that limit to the said heart. Thus in § 5.3, our analysis of the heart $\Psi_\nu\, \mathit{coh}\, W$ (associated to the birational model $W = \nu X$) is greatly simplified by studying the poset $\Psi_\nu\left[\mathit{per}\left(\frac{W}{Z}\right), \mathit{coh}\, W\right]_{\text{alg}}$ which has the distinct advantage of being a collection of algebraic hearts defined convex-geometrically, and is thus being readily tracked under mutation functors. In this section we instead study families of t-structures which sit in the Picard–group orbit of the reference heart $\mathsf{H} = \mathit{per}\left(\frac{X}{Y}\right)$, shedding light on the global structure of the partial order on $\operatorname{tilt}(\mathsf{H})$.

To spell this out, note that tensoring by any line bundle $\mathcal{L} \in \operatorname{Pic} X$ naturally gives an autoequivalence of $\mathbf{D}^b X$. The functor preserves support and evidently restricts to an autoequivalence of $\mathbf{D}^0 X$, whence we have an action of $\operatorname{Pic} X$ on $\mathbf{D}^0 X$. We focus on the induced action on the poset of t-structures $\text{t-str}(\mathbf{D}^0 X)$, where $\mathcal{L} \in \operatorname{Pic} X$ thus twists the heart K to

$$\mathcal{L} \otimes \mathsf{K} = \{\mathcal{L} \otimes k \mid k \in \mathsf{K}\}.$$

In what follows we study the orbit of H in relation to the poset $\operatorname{tilt}(\mathsf{H})$, setting up notation and stating key results. Before proving these results in § 6.3, we examine the most important consequences– the classifications of intermediate t-structures and bricks– in §§ 6.1 and 6.2 respectively.

The action of a single line bundle. Certain twists can be described explicitly– namely, if we write $\{\mathcal{L}_i \mid i \in \Delta \backslash \mathfrak{J}\}$ for the standard basis of $\operatorname{Pic} X$ (i.e. the line bundle $\mathcal{L}_i$ has degree 1 on $C_i \subset X$ and is trivial on other exceptional curves), then we have the following.

Proposition 6.1. *Fix* $i \in \Delta \setminus \mathfrak{J}$ *and* $n > 0$, *and consider the line bundle* $\mathcal{L} = \mathcal{L}_i^{\otimes n}$. *The following statements hold.*

(1) The heart $\mathcal{L}^\vee \otimes \mathsf{H}$ *lies in* $\mathsf{H}[-1, 0]$, *and is the tilt of* H *in the smallest torsion class containing* $\mathcal{O}_{C_i}(-n-1)[1]$.

(2) The heart $\mathcal{L} \otimes \mathsf{H}[-1]$ *lies in* $\mathsf{H}[-1, 0]$, *and is the tilt of* H *in the smallest torsion-free class containing* $\mathcal{O}_{C_i}(n-2)$.

If we denote by $X \to X_i$ the crepant partial contraction of $\bigcup_{j \in \Delta \setminus \{i\}} C_i$ (i.e. all exceptional curves *except* C_i), then it is clear from the above description that we have chains of inequalities

$$\mathsf{H} \;>\; \mathcal{L}_i^\vee \otimes \mathsf{H} \;>\; (\mathcal{L}_i^\vee)^{\otimes 2} \otimes \mathsf{H} \;> \dots >\; \mathit{per}\left(\tfrac{X}{X_i}\right) \;>\; \overline{\mathit{per}}\left(\tfrac{X}{X_i}\right) \;> \dots >\; \mathcal{L}_i^{\otimes 2} \otimes \mathsf{H}[-1] \;>\; \mathcal{L}_i \otimes \mathsf{H}[-1] \;>\; \mathsf{H}[-1].$$

Proposition 6.2. *In* H, *the torsion class* $\mathsf{H} \cap \mathit{per}\left(\frac{X}{X_i}\right)[1]$ *is generated by the set* $\{\mathcal{O}_{C_i}(n)[1] \mid n \leqslant -2\}$ *while the torsion-free class* $\mathsf{H} \cap \overline{\mathit{per}}\left(\frac{X}{X_i}\right)$ *is generated by* $\{\mathcal{O}_{C_i}(n) \mid n \geqslant -1\}$. *In other words, we have*

$$\mathit{per}\left(\tfrac{X}{X_i}\right) = \inf\left\{(\mathcal{L}_i^\vee)^{\otimes n} \otimes \mathsf{H} \mid n \geqslant 0\right\}, \qquad \overline{\mathit{per}}\left(\tfrac{X}{X_i}\right) = \sup\left\{\mathcal{L}_i^{\otimes n} \otimes \mathsf{H}[-1] \mid n \geqslant 0\right\}.$$

The actions of multiple nef monoids. In fact we prove a vast generalisation of proposition 6.2, realising geometric and semi-geometric hearts as limits of twists by line bundles on birational models. To explain this, note that if $W = \nu X$ is a birational model then the natural action of $\operatorname{Pic} W$ on $\mathbf{D}^0 W$ can be translated across the flop functor Ψ_ν to an action on $\mathbf{D}^0 X$. For brevity we continue to denote the action on $\mathbf{D}^0 X$ by $\otimes$, i.e. for $\mathcal{L} \in \operatorname{Pic} W$ and $h \in \mathbf{D}^0 X$ we have

$$\mathcal{L} \otimes h = \Psi_\nu\left(\mathcal{L} \otimes \Psi_\nu^{-1} h\right)$$

where the tensor product on the right-hand side is over $\mathcal{O}_W$.

Now $\operatorname{Pic} W$ sits as the lattice of integral points in the vector space $\operatorname{Pic}_{\mathbb{R}} W$, which we identify (as in proposition 5.5) with $\operatorname{Pic}_{\mathbb{R}} X \cong \{\theta \in \Theta(\underline{\Delta} \setminus \mathfrak{J}) \mid \theta(\delta_{\mathfrak{J}}) = 0\}$ by taking proper transforms across the birational map $\rho : W \dashrightarrow X$. Thus each cone $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$ determines a monoid $\operatorname{Pic} W \cap \rho^* \sigma$ which contains line bundles on W whose proper transform to X lies in σ. We simply write $\operatorname{Pic} W \cap \sigma$ for this monoid, leaving the birational map ρ implicit.

Given a cone $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$ we can always choose a birational model $W = \nu X$ such that the monoid $\operatorname{Pic} W \cap \sigma$ lies in the nef cone $\operatorname{Pic}^+ W$. We say such a birational model W is *σ-positive*, and the condition is equivalent to saying σ lies in the heart cone $\mathbf{C}(\Psi_\nu \operatorname{coh} W)$. For such σ and W we consider the $(\operatorname{Pic} W \cap \sigma)$-orbit of the reference heart $H = \operatorname{per}(\frac{X}{Y})$.

The following result then shows that the orbit is independent of the choice of σ-positive birational model, lies in $\operatorname{tilt}(H)$, and limits to the (semi-)geometric heart determined by σ.

Theorem 6.3. *Given a sequence of flops ν from X with corresponding birational model $W = \nu X$, the following statements hold in $\mathbf{D}^0 X$.*

(1) Let $\mathcal{L} \in \operatorname{Pic} W$ be a line bundle. Then the heart $\mathcal{L} \otimes H$ is intermediate with respect to H if and only if $\mathcal{L}^\vee$ is nef, and in this case $\mathcal{L} \otimes H$ lies in the interval $[\Psi_\nu \operatorname{coh} W, H]$.

Likewise the heart $\mathcal{L} \otimes H[-1]$ is intermediate with respect to H if and only if $\mathcal{L}$ is nef, and in this case $\mathcal{L} \otimes H[-1]$ lies in the interval $[H[-1], \Psi_\nu \overline{\operatorname{coh}} W]$.

(2) Let $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$ be a a cone such that W is σ-positive, i.e. a face of $\mathbf{C}(\Psi_\nu \operatorname{coh} W)$. Then

$$H^{tt}(\sigma) = \inf \{\mathcal{L}^\vee \otimes H \mid \mathcal{L} \in \operatorname{Pic} W \cap \sigma\}, \qquad H_{tt}(\sigma) = \sup \{\mathcal{L} \otimes H[-1] \mid \mathcal{L} \in \operatorname{Pic} W \cap \sigma\}.$$

Further, suppose $W' \in \operatorname{Bir}(\frac{X}{Z})$ is another birational model, and $\mathcal{L}' \in \operatorname{Pic} W'$ is a line bundle with proper transform $\mathcal{L} \in \operatorname{Pic} W$. Then the following statements hold.

(3) If $\mathcal{L}'^\vee \otimes H \in [\Psi_\nu \operatorname{coh} W, H]$ or $\mathcal{L}' \otimes H \in [H[-1], \Psi_\nu \overline{\operatorname{coh}} W]$, then $\mathcal{L}$ is nef on W.

(4) If the hearts $\mathcal{L} \otimes H$ and $\mathcal{L}' \otimes H$ are both intermediate with respect to H, then they are in fact equal i.e. we have an equality of t-structures $\mathcal{L} \otimes H = \mathcal{L}' \otimes H$.

Likewise if the hearts $\mathcal{L} \otimes H[-1]$ and $\mathcal{L}' \otimes H[-1]$ are both intermediate with respect to H, then they are equal.

Thus the limit of all twists $\mathcal{L}^\vee \otimes H$ ranging over $\mathcal{L} \in \operatorname{Pic}^+ W$ is the geometric heart $\Psi_\nu \operatorname{coh} W$, which is fixed by the action of any $\mathcal{L} \in \operatorname{Pic} W$. More generally given any non-zero cone $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$, corollary 5.29 gives a birational model $W = \nu X$ and a partial contraction $W \to Y$ such that $H^{tt}(\sigma) = \Psi_\nu \operatorname{per}(\frac{W}{Y})$. Then the monoid $\operatorname{Pic} W \cap \sigma$ can be naturally identified with $\operatorname{Pic}^+ Y$, and the limit of the twists $\mathcal{L}^\vee \otimes H$ ranging over $\mathcal{L} \in \operatorname{Pic} W \cap \sigma$ is precisely the heart $\operatorname{per}(\frac{W}{Y})$ which is geometric on the curves 'detected by' $\operatorname{Pic} W \cap \sigma$.

Global structure of the order. Theorem 6.3 (4) in shows that the sub-poset of $\operatorname{alg-tilt}(H)$ given by

$$\left\{\mathcal{L}^\vee \otimes H \mid \mathcal{L} \in \operatorname{Pic}^+ W \text{ for some } W \in \operatorname{Bir}(\tfrac{X}{Z})\right\} \sqcup \left\{\mathcal{L} \otimes H[-1] \mid \mathcal{L} \in \operatorname{Pic}^+ W \text{ for some } W \in \operatorname{Bir}(\tfrac{X}{Z})\right\} \tag{25}$$

is naturally in bijection with $\operatorname{Pic} X \sqcup \operatorname{Pic} X$ and decomposes in accordance with the movable fan (17). Analysing $\operatorname{alg-tilt}(H)$ in comparison with this subposet makes its highly regular structure more transparent.

To set notation, suppose $W = \nu X$ is a birational model of X with exceptional curves $C_i \subset W$ $(i \in \Delta \setminus \nu\mathfrak{J})$. The intersection pairing with these 1-cycles gives an isomorphism $\deg : \operatorname{Pic} W \to \mathbb{Z}^{\Delta \setminus \nu\mathfrak{J}}$ which assigns $\mathcal{L} \in \operatorname{Pic} W$ to the integer tuple $\deg \mathcal{L} = ((\mathcal{L} \cdot C_i))_{i \in \Delta \setminus \nu\mathfrak{J}}$.

Write $\underline{0}, \underline{1} \in \mathbb{Z}^{\Delta \setminus \nu\mathfrak{J}}$ for the tuples whose entries are all 0 or all 1 respectively.

We endow $\mathbb{Z}^{\Delta \setminus \nu\mathfrak{J}}$ with the product order, i.e. for tuples $(a_i), (b_i) \in \mathbb{Z}^{\Delta \setminus \nu\mathfrak{J}}$ we have $(a_i) \leqslant (b_i)$ if and only if $a_i \leqslant b_i$ for each $i \in \Delta \setminus \nu\mathfrak{J}$. Thus for example a line bundle $\mathcal{L} \in \operatorname{Pic} W$ is nef if and only if $\underline{0} \leqslant \deg \mathcal{L}$, and two line bundles $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic} W$ satisfy $\deg \mathcal{L} \leqslant \deg \mathcal{L}'$ if and only if $\mathcal{L}' \otimes \mathcal{L}^\vee$ is nef.

The partial order on various twists and shifts of H can then be described as follows.

Corollary 6.4. *Given line bundles* $\mathscr{L}, \mathscr{L}'$ *on some birational model* W *of* X *and integers* $i, j \in \mathbb{Z}$, *we have* $\mathscr{L} \otimes \mathsf{H}[i] \geqslant \mathscr{L}' \otimes \mathsf{H}[j]$ *in* $\text{t-str}(\mathbf{D}^0 X)$ *if and only if either* $i \geqslant j$, *or* $i = j$ *and* $\deg \mathscr{L} \geqslant \deg \mathscr{L}'$.

Thus the poset $\{\mathscr{L} \otimes \mathsf{H}[i] \mid \mathscr{L} \in \operatorname{Pic} W,\ i \in \mathbb{Z}\}$ *is isomorphic to* $\mathbb{Z} \times \mathbb{Z}^{|\Delta \setminus \mathfrak{J}|}$ *with the lexicographic order.*

Proof. If $i = j$, then it is clear that $\mathscr{L} \otimes \mathsf{H}[i] \geqslant \mathscr{L}' \otimes \mathsf{H}[j]$ if and only if $\mathsf{H} \geqslant \mathscr{L}' \otimes \mathscr{L}^{\vee} \otimes \mathsf{H}$, which by theorem 6.3 (1) occurs if and only if $\mathscr{L} \otimes \mathscr{L}'^{\vee}$ is nef i.e. $\deg \mathscr{L} \geqslant \deg \mathscr{L}'$.

Thus to conclude it suffices to show $\mathscr{L} \otimes \mathsf{H} > \mathscr{L}' \otimes \mathsf{H}[-1]$ for all $\mathscr{L}, \mathscr{L}' \in \operatorname{Pic} W$. Now we may write $\mathscr{L} = \mathscr{L}_+ \otimes \mathscr{L}_-^{\vee}$ where the line bundles $\mathscr{L}_+, \mathscr{L}_- \in \operatorname{Pic} W$ are nef. Theorem 6.3 (1) shows there are inequalities $\mathscr{L}_+ \otimes \mathsf{H}[-1] \geqslant \mathsf{H}[-1]$ and $\mathscr{L}_-^{\vee} \otimes \mathsf{H} > \Psi_v \operatorname{coh} W$. Applying the functor $\mathscr{L}_-^{\vee} \otimes (-)[1]$ to the first inequality thus yields $\mathscr{L} \otimes \mathsf{H} > \Psi_v \operatorname{coh} W$, and a similar argument gives $\Psi_v \overline{\operatorname{coh}} W > \mathscr{L}' \otimes \mathsf{H}[-1]$ from which the conclusion follows. ☐

The poset (25) thus decomposes as a union of sub-posets

$$\bigcup_{W \in \operatorname{Bir}(\frac{X}{Z})} \left(\{\mathscr{L}^{\vee} \otimes \mathsf{H} \mid \mathscr{L} \in \operatorname{Pic}^+ W\} \sqcup \{\mathscr{L} \otimes \mathsf{H}[-1] \mid \mathscr{L} \in \operatorname{Pic}^+ W\} \right),$$

and corollary 6.4 let us read off the order in each component of the decomposition. It can be shown that there are no further relations, i.e. if two elements of (25) are comparable then the relation necessarily arises in one of the ways described in corollary 6.4 for some birational model W.

Finally, we compare the above orbits of H with other intermediate algebraic hearts.

Theorem 6.5. *For any t-structure* $\mathsf{K} \in \operatorname{tilt}^+(\mathsf{H})$, *there is a birational model* W *of* X *and line bundles* $\mathscr{L}, \mathscr{L}' \in \operatorname{Pic} W$ *such that*

$$\underline{0} \geqslant \deg \mathscr{L} > \deg \mathscr{L}' \geqslant \deg \mathscr{L} - \underline{1}, \qquad \textit{and} \qquad \mathscr{L} \otimes \mathsf{H} \geqslant \mathsf{K} > \mathscr{L}' \otimes \mathsf{H}.$$

Likewise for any $\mathsf{K} \in \operatorname{tilt}^-(\mathsf{H})$, *there is a birational model* W *of* X *and line bundles* $\mathscr{L}, \mathscr{L}' \in \operatorname{Pic} W$ *such that*

$$\underline{0} \leqslant \deg \mathscr{L} < \deg \mathscr{L}' \leqslant \deg \mathscr{L} + \underline{1}, \qquad \textit{and} \qquad \mathscr{L} \otimes \mathsf{H}[-1] \leqslant \mathsf{K} < \mathscr{L}' \otimes \mathsf{H}[-1].$$

§ 6.1 Every torsion theory on the perverse heart is numerical. The above analysis of $\operatorname{tilt}(\mathsf{H})$ paves the way for this result, showing that the heart fan of H detects *every* intermediate t-structure and thus an arbitrary heart in $\operatorname{tilt}(\mathsf{H})$ must be one of the algebraic, geometric, or semi-geometric t-structures described in §§ 3 to 5.

Theorem 6.6. *Let* $\mathsf{K} \in \operatorname{tilt}(\mathsf{H})$ *be an arbitrary tilt of* H. *Then the heart cone* $\mathbf{C}(\mathsf{K})$ *is non-zero.*

We prove theorem 6.6 over the course of this subsection. Thus, fix an arbitrary $\mathsf{K} \in \operatorname{tilt}(\mathsf{H})$. If K is algebraic, then $\mathbf{C}(\mathsf{K})$ is full-dimensional (in particular, non-zero) by theorems 4.1 and 4.20 so we may assume K is non-algebraic. One then expects $\mathbf{C}(\mathsf{K})$ to be a cone in $\operatorname{Arr}(\Delta, \mathfrak{J})$, and the following lemma is the key tool we exploit.

Lemma 6.7. *Given* $\mathsf{K} \in \operatorname{tilt}(\mathsf{H})$, *suppose there is a non-zero cone* $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$ *and a* σ*-positive birational model* W *such that* $\mathscr{L}^{\vee} \otimes \mathsf{H} \geqslant \mathsf{K} \geqslant \mathscr{L} \otimes \mathsf{H}[-1]$ *for every* $\mathscr{L} \in \operatorname{Pic} W \cap \sigma$. *Then the heart cone* $\mathbf{C}(\mathsf{K})$ *contains* σ *and is in particular non-zero.*

Proof. The given conditions combined with theorem 6.3 (2) show $\mathsf{H}^{\mathrm{tt}}(\sigma) \geqslant \mathsf{K} \geqslant \mathsf{H}_{\mathrm{tt}}(\sigma)$, so the result follows from lemma 2.12. ☐

It is relatively straightforward to obtain the bound on one side for lemma 6.7.

Lemma 6.8. *If* $K \in \operatorname{tilt}(H)$ *is not algebraic, then there is a non-zero cone* $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$ *and a* σ*-positive birational model* W *such that* $\mathcal{L}^\vee \otimes H \geqslant K$ *for every* $\mathcal{L} \in \operatorname{Pic} W \cap \sigma$.

Likewise, there is a (possibly different) non-zero cone $\sigma' \in \operatorname{Arr}(\Delta, \mathfrak{J})$ *and a* σ'*-positive birational model* W' *such that* $K \geqslant \mathcal{L}' \otimes H[-1]$ *for every* $\mathcal{L}' \in \operatorname{Pic} W' \cap \sigma'$.

Proof. We prove the first statement, the second being analogous. Since K is non-algebraic, corollary 4.7 shows that K is not covered by an algebraic heart i.e. if $K_n \in \text{alg-tilt}(H)$ satisfies $K_n > K$ then there is an algebraic heart K_{n+1} satisfying $K_n \gtrdot K_{n+1} > K$. Starting with the tautological relation $H = K_0 > K$, this produces an infinite chain $K_0 > K_1 > K_2 > \dots > K$ approaching K from above using elements of $\operatorname{tilt}^+(H)$.

By theorem 6.5, for each K_n there is a birational model W_n and line bundles $\mathcal{L}_n, \mathcal{L}'_n \in \operatorname{Pic} W_n$ such that $\mathcal{L}_n \otimes H \geqslant K_n \geqslant \mathcal{L}'_n \otimes H$ and $\underline{0} \geqslant \deg \mathcal{L}_n > \deg \mathcal{L}'_n \geqslant \deg \mathcal{L}_n - \underline{1}$. But there are only finitely many birational models of X, so we can pass to a subsequence and reindex if necessary to assume all birational models W_n are equal. Thus there is a $W = \nu X \in \operatorname{Bir}\left(\frac{X}{Z}\right)$ and a sequence of anti-nef line bundles $\mathcal{L}_1, \mathcal{L}_2, \dots \in \operatorname{Pic} W$ such that $\mathcal{L}_n \otimes H > K$ for each n.

Claim there is some integral exceptional curve $C_i \subset W$ such that $(\mathcal{L}_n \cdot C_i)$ attains arbitrarily large magnitude, i.e. for any $N < 0$ there is some n with $(\mathcal{L}_n \cdot C_i) < N$. If not, we can choose a line bundle $\mathcal{L}_\infty \in \operatorname{Pic} W$ such that $\deg \mathcal{L}_\infty \leqslant \deg \mathcal{L}_n - \underline{1}$ for each n. Thus $\deg \mathcal{L}'_n \geqslant \deg \mathcal{L}_\infty$ for each n, which shows $\mathcal{L}'_n \otimes H \geqslant \mathcal{L}_\infty \otimes H$ and hence each K_n lies in the interval $[\mathcal{L}_\infty \otimes H, H]$. But this contradicts theorem 4.1 (3), which states that intervals in $\operatorname{tilt}^+(H)$ are finite.

Writing $\mathcal{L} \in \operatorname{Pic} W$ for the line bundle which has degree 1 on C_i and is trivial elsewhere, we therefore see that for every $N < 0$ there is an n such that $\mathcal{L}^{\otimes N} \otimes H > \mathcal{L}_n \otimes H$ and thus $\mathcal{L}^{\otimes N} \otimes H > K$. Choosing $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J}) \cong \operatorname{Mov}(X)$ to be the ray generated by (the proper transform of) $\mathcal{L}$ then yields the result. □

As an immediate consequence, we see that any t-structure bounded by a geometric heart is numerical.

Lemma 6.9. *If* $K \in \operatorname{tilt}(H)$ *is such that* $K \geqslant \Psi_\nu \operatorname{\mathcal{coh}} W$ *or* $\Psi_\nu \overline{\operatorname{\mathcal{coh}}} W \geqslant K$ *for some birational model* $W = \nu X$*, then* $\mathbf{C}(K)$ *is non-zero.*

Proof. Again we may assume K is non-algebraic and lies in $[\Psi_\nu \operatorname{\mathcal{coh}} W, H]$, so that lemma 6.8 gives a non-zero cone $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$ and a σ-positive birational model W' such that $\mathcal{L}'^\vee \otimes H \geqslant K \geqslant \Psi_\nu \operatorname{\mathcal{coh}} W$ for every $\mathcal{L}' \in \operatorname{Pic} W' \cap \sigma$. But then theorem 6.3 (3) shows that for each $\mathcal{L}' \in \operatorname{Pic} W' \cap \sigma$, the proper transform $\mathcal{L} \in \operatorname{Pic} W$ is also nef and by theorem 6.3 (4) satisfies $\mathcal{L}'^\vee \otimes H = \mathcal{L}^\vee \otimes H$. In particular, W is σ-positive and we have $\mathcal{L}^\vee \otimes H \geqslant K$ for every $\mathcal{L} \in \operatorname{Pic} W \cap \sigma$. On the other hand we also have the inequality $K \geqslant \Psi_\nu \operatorname{\mathcal{coh}} W \geqslant \mathcal{L} \otimes H[-1]$ for every $\mathcal{L} \in \operatorname{Pic} W \cap \sigma$, so lemma 6.7 yields the conclusion. □

We deduce analogous bounds on hearts $K \in \operatorname{tilt}(H)$ which contain skyscraper sheaves $\mathcal{O}_p$ (for $p \in X$) or shifts thereof, noting that a priori each $\mathcal{O}_p$ is only a two-term complex in $K[0, 1]$.

Lemma 6.10. *Suppose* $C_i \subset X$ *is an integral exceptional curve with a closed point* $p \in C_i$*, and* $\mathcal{L}_i \in \operatorname{Pic} X$ *is the line bundle which has degree* 1 *on* C_i *and is trivial elsewhere. Writing* $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$ *for the cone generated by* $\mathcal{L}_i$*, the following statements hold for any* $K \in \operatorname{tilt}(H)$.

(1) If $\mathcal{O}_p \in K$*, then* $K \geqslant \mathcal{L} \otimes H[-1]$ *for every* $\mathcal{L} \in \operatorname{Pic} X \cap \sigma$.

(2) If $\mathcal{O}_p \in K[1]$*, then* $\mathcal{L}^\vee \otimes H \geqslant K$ *for every* $\mathcal{L} \in \operatorname{Pic} X \cap \sigma$.

In particular if there are closed points $p, q \in C_i$ *such that* $\mathcal{O}_p \in K$ *and* $\mathcal{O}_q \in K[1]$*, then* $\mathbf{C}(K)$ *is non-zero.*

Proof. If $p \in C_i$ is a closed point such that $\mathcal{O}_p \in K$, then $\mathcal{O}_p$ lies in the torsion-free class $F = K \cap H$. The exact triangle $\mathcal{O}_{C_i}(-1) \to \mathcal{O}_p \to \mathcal{O}_{C_i}(-2)[1] \to \mathcal{O}_{C_i}(-1)[1]$ shows $\mathcal{O}_{C_i}(-1)$ is a sub-object (in H) of $\mathcal{O}_p$, thus we also have $\mathcal{O}_{C_i}(-1) \in F$. Further, inductively considering extensions $\mathcal{O}_{C_i}(n) \to \mathcal{O}_{C_i}(n+1) \to \mathcal{O}_p \to \mathcal{O}_{C_i}(n)[1]$ shows

$\mathcal{O}_{C_i}(n) \in F$ for every $n \geqslant -1$, and thus F contains each torsion class $\mathcal{L}_i^{\otimes n} \otimes H[-1] \cap H$ by proposition 6.1. Statement (1) follows, and the proof of (2) is analogous. □

Thus given an arbitrary non-algebraic t-structure K, we show $\mathbf{C}(K)$ is non-zero by comparing K against some appropriately chosen geometry. The following lemma informs this choice.

Lemma 6.11. *Given* $K \in \mathrm{tilt}(H)$*, there is a sequence of flops* ν *from* X *with corresponding birational model* $W = \nu X$ *such that* K *lies in* $\Psi_\nu\, \mathit{per}(\frac{W}{Z})[-1,0]$ *and contains the objects* $\Psi_\nu \mathcal{O}_{C_i}(-1)$ *for each exceptional curve* $C_i \subset W$.

Proof. Consider the sub-poset of tilt(H) given by

$$[K, H] \cap \{\Psi_\nu H \mid \nu \text{ a spherical } \mathfrak{J}\text{-path}\},$$

where we use the notational shorthand $\Psi_\nu H = \Psi_\nu\, (\mathit{flmod}_\nu \Lambda_\nu)$ as in § 3.4 and continue to make the identification $\mathit{flmod}_\nu \Lambda_\nu = \mathit{per}(\frac{\nu X}{Z})$ as in § 5.1. The above poset is non-empty (it contains H) and finite, so has a minimal element determined by some spherical $\mathfrak{J}$-path ν. This determines our sequence of flops ν. Evidently we have $H \geqslant \Psi_\nu H \geqslant K$ and $K \geqslant H[-1] \geqslant \Psi_\nu H[-1]$, i.e. K lies in $\Psi_\nu H[-1, 0]$.

Now the heart $\Psi_\nu H$ lies in $H[-1, 0]$ and $K[0, 1]$. Suppose there is an $i \in \Delta \setminus \nu\mathfrak{J}$ such that K does not contain $\Psi_\nu S_i = \Psi_\nu \mathcal{O}_{C_i}(-1)$. Then $\Psi_\nu S_i$, being simple in $\Psi_\nu H$, necessarily lies in $K[1]$ and thus in H since $H[-1] \cap K[1] = 0$. In particular we have $\Psi_\nu S_i \in V_\nu = H \cap \Psi_\nu$, and thus (by theorem 4.6) the inequality $H \geqslant \Psi_\nu H > \Psi_{\nu_i \nu} H$.

But we also have $\Psi_{\nu_i\nu} \geqslant K$, since the torsion class $K[1] \cap H$ contains $U_\nu = \Psi_\nu H[1] \cap H$ and $\Psi_\nu S_i$ and hence also (by the same proposition) $U_{\nu_i\nu} = \Psi_{\nu_i\nu} H[1] \cap H$. This contradicts the minimality of $\Psi_\nu H$, so K must contain every simple $\Psi_\nu S_i$ for $i \in \Delta \setminus \nu\mathfrak{J}$. □

Say a t-structure $K \in$ t-str$(\mathbf{D}^0 X)$ *lives on* X if there are inclusions $\langle \mathcal{O}_{C_i}(-1) \mid i \in \Delta \setminus \mathfrak{J} \rangle \subseteq K \subseteq \mathit{per}(\frac{X}{Z})[-1, 0]$, i.e. the sequence of flops for K given by the above lemma is trivial. Then lemma 6.11 explains that up to the application of a flop functor, it suffices to work only with those $K \in \mathrm{tilt}(H)$ which live on X.

This additional hypothesis gives us tremendous control on the torsion pair associated to K, in particular we show that every skyscraper on X is either torsion or torsion-free.

Lemma 6.12. *If* $K \in \mathrm{tilt}(H)$ *lives on* X *and* $p \in X$ *is a closed point in the exceptional locus, then* $\mathcal{O}_p$ *lies in a single cohomological degree with respect to* K *i.e. we have* $\mathcal{O}_p \in K$ *or* $\mathcal{O}_p \in K[1]$.

Proof. The intermediate heart K induces the torsion pair $H = (K[1] \cap H) * (K \cap H)$, and thus a filtration $t \to \mathcal{O}_p \to f \to t[1]$ with $t \in K[1] \cap H$, $f \in K \cap H$. In particular, considering $\mathbf{K}$-classes $[t] = \sum_{i \in \Delta \setminus \mathfrak{J}} t_i \alpha_i$ and $[f] = \sum_{i \in \Delta \setminus \mathfrak{J}} f_i \alpha_i$, we see that for each i the t_i, f_i are non-negative integers satisfying $t_i + f_i = \delta_i$.

Here the coefficients δ_i of $\delta_{\mathfrak{J}} = [\mathcal{O}_p]$ correspond to ranks of the summands of $\mathcal{V}(\frac{X}{Z}) = \bigoplus_{i \in \Delta \setminus \mathfrak{J}} \mathcal{N}_i$ as in lemma 5.6. In particular we have $\delta_0 = \mathrm{rk}(\mathcal{N}_0) = 1$ (since $\mathcal{N}_0 = \mathcal{O}_X$ by construction), so we must have $(t_0, f_0) = (0, 1)$ or $(t_0, f_0) = (1, 0)$.

If $f_0 = 0$, then f lies in the category $\langle S_i \mid i \in \Delta \setminus \mathfrak{J} \rangle$ but this implies $\mathrm{Hom}(\mathcal{O}_p, f) = 0$ since each $S_i = \mathcal{O}_{C_i}(-1)$ lies in $\mathcal{O}_p{}^{\perp}$. Thus the map $t \to \mathcal{O}_p$ is an isomorphism and $\mathcal{O}_p$ lies in $K[1]$.

On the other hand if $t_0 = 0$, then t lies in the category $\langle S_i \mid i \in \Delta \setminus \mathfrak{J} \rangle$, and hence in $K \cap H$ (since K lives on X). Thus in fact $t = 0$, and $\mathcal{O}_p = f \in K$. □

This gives us the necessary ingredients to show that an arbitrary t-structure in tilt(H) is numerical.

Proof of theorem 6.6. We may assume K is non-algebraic and, applying a flop functor if necessary, lives on X. Thus by lemma 6.12, for every closed point p in the exceptional locus $C = \bigcup_{i \in \Delta \setminus \mathfrak{J}} C_i \subset X$, either $\mathcal{O}_p$ or $\mathcal{O}_p[-1]$ lies in K.

If there is some curve C_i with two points $p, q \in C_i$ such that $\mathcal{O}_p \in K$ and $\mathcal{O}_q[-1] \in K$, then we obtain the conclusion using lemma 6.10.

Otherwise, since C is connected, either K or $K[1]$ contains the subcategory $\{\mathcal{O}_p \mid p \in C\}$. If $\mathcal{O}_p \in K$ for every $p \in C$, then (by lemma 6.10) we have $K \geqslant \mathcal{L} \otimes H[-1]$ for every $\mathcal{L} \in \mathrm{Pic}^+ X$ and thus $K \geqslant \overline{\mathit{coh}}\, X$. Further, the torsion-free part $K \cap H$ contains $H_{ss}(C^0_{\mathfrak{J}}) = \langle \mathcal{O}_p \mid p \in C\rangle$ so in fact $K \geqslant \mathit{coh}\, X$ and thus $\mathbf{C}(K) \neq 0$ by lemma 6.9. The case when $\mathcal{O}_p[-1] \in K$ for every $p \in C$ is handled likewise. ☐

§ 6.2 Classification of bricks. Using the above classification of t-structures we now show that every brick in $H = \mathit{per}(\frac{X}{Z})$ must arise from an algebraic or geometric simple.

Theorem 6.13. *Let* $b \in H$ *be a brick. Then either*

(1) there is a spherical $\mathfrak{J}$*-path* ν *and a closed point* $p \in \nu X$ *such that* $b = \Psi_\nu \mathcal{O}_p$*, or*

(2) there is a semibrick $S \subset H$ *that generates a functorially finite torsion class and contains* b*. In this case, there is a* $\mathfrak{J}$*-path* ν *and an index* $i \in \underline{\Delta} \setminus \nu\mathfrak{J}$ *such that* $b \in \{\Psi_\nu S_i[1], \Phi_\nu S_i\}$*.*

In particular, the **K***-theory class* $[b] \in \mathfrak{h}(\underline{\Delta} \setminus \mathfrak{J})$ *is a primitive restricted root of the form* $\varphi_\nu \alpha_i$ *or* $\delta_{\mathfrak{J}}$*.*

We prove the theorem over the course of this subsection. Accordingly, fix a brick $b \in H$ and consider the torsion pair $H = T * F$ where T is the minimal torsion class containing b. Write $K = F * T[-1]$ for the tilt, and note that $b[-1]$ is a simple of K (proposition 2.6).

If K is geometric (i.e. $K \in \Psi_\nu[\overline{\mathit{coh}}\, W, \mathit{coh}\, W]$ for some birational model $W = \nu X$), then the simples of K are of the form $\Psi_\nu \mathcal{O}_p$ or $\Psi_\nu \mathcal{O}_p[-1]$ for closed points $p \in W$. It follows that b must be of the form $\Psi_\nu \mathcal{O}_p$.

On the other hand if K is algebraic, then T is a functorially finite torsion class generated by the semibrick $\{b\}$ as required. The heart K must be given by $\Psi_\nu H$ or $\Phi_\nu H[-1]$ for some $\mathfrak{J}$-path ν, and it follows that every simple of K (in particular, $b[-1]$) is of the form $\Psi_\nu S_i$ or $\Phi_\nu S_i[-1]$ as required.

Thus it remains to consider the case when K is semi-geometric. Now Lemma 6.11 gives a spherical $\mathfrak{J}$-path υ such that $\Psi_\upsilon^{-1} K$ lives on W. In particular K is a tilt of $\Psi_\upsilon H$, and the corresponding torsion class $T \cap V_\upsilon \in \mathrm{tors}(\Psi_\upsilon H)$ is minimal containing b. Hence replacing K by $\Psi_\upsilon^{-1} K$ if necessary, we may assume K is a semi-geometric heart that lives on X.

It follows that $\mathbf{C}\, K = \mathbf{C}(\mathit{per}(\frac{X}{Y}))$ for some partial contraction $\tau : X \to Y$, which contracts the curves $C_I = \bigcup_{i \in I} C_i$ determined by a non-empty proper subset $I \subset \Delta \setminus \mathfrak{J}$. Considering the subcategory $\mathbf{D}^0 X_I \subset \mathbf{D}^0 X$ containing complexes supported on C_I, we study the restriction $K_I = K \cap \mathbf{D}^0 X_I$ in relation to $H_I = \mathit{per}(\frac{X}{Z}) \cap \mathbf{D}^0 X_I$.

Lemma 6.14. *The category* K_I *is an algebraic tilt of* H_I*.*

Proof. That K_I is a tilt of H_I follows from theorem 5.27. If K_I weren't algebraic, then K would be geometric on a larger number of curves than $\mathit{per}(\frac{X}{Y})$, whence $\mathbf{C}(K)$ would be a larger-dimensional cone than $\mathbf{C}(\mathit{per}(\frac{X}{Y}))$, a contradiction. ☐

Thus every simple object of K is either one of the finitely many simples of K_I, or of the form $\mathcal{O}_p$ or $\mathcal{O}_p[-1]$ for closed points $p \in X \setminus C_I$. If $b[-1] = \mathcal{O}_p[-1]$ we are again done, so we only need to consider the case when $b[-1]$ is a simple of K_I. Note this guarantees that C_I is connected and b is supported on every point of C_I.

Lemma 6.15. *Under the above assumptions, we have* $\mathcal{O}_p \in T$ *whenever* $p \in C_I$*.*

Proof. Consider a curve C_i that is *not* contracted by τ, so that the restriction of K to C_i is geometric. Thus K contains the sheaf $k = \mathcal{O}_{C_i}(-1)$, which has some simple sub-object $s \in K$ (proposition 2.6).

If s is supported in $X \setminus C_I$, then $s \neq b$ and thus s lies in H. This is only possible if $s = \mathcal{O}_p$ for some $p \in X \setminus C_I$, but in that case $\mathrm{Hom}(s, k) = 0$ which is a contradiction.

It follows that s is a simple of K_I, and considering supports shows the inclusion $s \to k$ must factor through the object $\mathcal{O}_p[-1]$ where p the unique closed point in $C_i \cap C_I$. Since K was chosen to live on X, $\mathcal{O}_p$ lies in a single cohomological degree with respect to K (lemma 6.12) and thus $\mathcal{O}_p[-1]$ must lie in K_I.

But K_I is an algebraic tilt of H_I, in particular the objects of K_I are determined by inequalities on $\mathbf{K}$-theory. It follows that for any $q \in C_I$, the object $\mathcal{O}_q$ cannot lie in K_I, and thus $\mathcal{O}_q[-1] \in K_I \subset K$ for every $q \in C_I$. The result follows. □

Lemma 6.16. *If* C_i *is a curve* not *contracted by* $\tau : X \to Y$*, then* $\operatorname{Hom}(\mathcal{O}_{C_i}(-1), b) = \operatorname{Hom}(b, \mathcal{O}_{C_i}(-1)) = 0$.

Proof. Consider the unique point $p \in C_i \cap C_I$. Since $\mathcal{O}_p$ lies in the torsion class T generated by b, there is a (necessarily non-split) surjection $b \twoheadrightarrow \mathcal{O}_p$ in H. Now any non-zero morphism $\mathcal{O}_{C_i}(-1) \to b$ would factor through $\mathcal{O}_p$, but this would split the surjection $b \twoheadrightarrow \mathcal{O}_p$, furnishing a contradiction. Thus $\operatorname{Hom}(\mathcal{O}_{C_i}(-1), b) = 0$. On the other hand, b evidently lies in $K[1]$ while $\mathcal{O}_{C_i}(-1)$ lies in K, thus $\operatorname{Hom}(b, \mathcal{O}_{C_i}(-1)) = 0$ too. □

By the above lemma, we thus have a semibrick

$$S = \{b\} \cup \{\mathcal{O}_{C_i}(-1) \mid i \in \Delta \setminus (\mathfrak{J} \cup I)\} \quad \subset H.$$

Write $T' \in \operatorname{tors}(H)$ for the torsion class generated by S, and K' for the corresponding tilt of H. We use this to conclude the proof of theorem 6.13.

Lemma 6.17. *The torsion class* T' *is functorially finite, i.e.* K' *is an algebraic tilt of* K.

Proof. Evidently $T' \cap \mathbf{D}^0 X_I = T \cap \mathbf{D}^0 X_I$, so that $K' \cap \mathbf{D}^0 X_I = K_I$.

If C_i is a curve not contracted by τ, then T contains the sheaf $\mathcal{O}_p$ for $p \in C_I \in C_i$, and thus it contains the quotient $\omega_{C_i}[1]$. Since T' also contains $\mathcal{O}_{C_i}(-1)$, it follows that T' in fact contains the full subcategory

$$\langle \omega_{C_i}[1], \mathcal{O}_{C_i}(-1) \rangle = \{x \in \mathit{per}(\tfrac{X}{Z}) \mid \operatorname{Supp}(x) \subset C_i\}.$$

In other words, K' restricts to the algebraic category $H[-1]$ on every curve C_i not contracted by τ. Since K' restricts to an algebraic category on C_I too, K' cannot be a (semi-)geometric tilt of H hence we are done. □

§ 6.3 Actions of Picard groups. We now repay the technical debt, and prove propositions 6.1 and 6.2 and theorems 6.3 and 6.5 by examining the actions of Picard groups on t-structures, heart cones, and modifying modules. This builds upon the work of Hirano–Wemyss [HW23, §7] on the subject.

Actions on the heart fan. The action of $\operatorname{Pic} X$ on the Grothendieck group $\mathbf{K}X$ is straightforward to analyse after choosing appropriate bases– on $\operatorname{Pic} X$ we consider the generators $\{\mathcal{L}_i \mid i \in \Delta \setminus \mathfrak{J}\}$ where $\mathcal{L}_i = (\det \mathcal{N}_i)^\vee$ has degree 1 on $C_i \subset X$ and is trivial on other exceptional curves. On the other hand, the classes $\delta_{\mathfrak{J}} = [\mathcal{O}_p]$ and $\alpha_i = [\mathcal{O}_{C_i}(-1)]$ $(i \in \Delta \setminus \mathfrak{J})$ give a basis for $\mathbf{K}X$, and then the action $\operatorname{Pic} X \circlearrowright \mathbf{K}X$ is given by

$$\mathcal{L}_i \otimes \delta_{\mathfrak{J}} = \delta_{\mathfrak{J}}, \qquad \mathcal{L}_i \otimes \alpha_j = \begin{cases} \alpha_j + \delta_{\mathfrak{J}}, & j = i \\ \alpha_j, & \text{otherwise} \end{cases}. \tag{26}$$

This clearly preserves the root system, i.e. the transposed action on $\Theta(\underline{\Delta} \setminus \mathfrak{J})$ given by $\mathcal{L} \cdot \theta = \theta \circ (\mathcal{L} \otimes -)$ preserves the intersection arrangement $\operatorname{Arr}(\underline{\Delta}, \mathfrak{J})$ and in particular takes chambers to chambers. This can be visualised by noting that the functionals $\{\alpha_i \mid i \in \Delta \setminus \mathfrak{J}\}$ give coordinates on the level set $\{\delta_{\mathfrak{J}} = 1\}$ and the action of $\operatorname{Pic} X$ to the level set restricts to translations along the lattice of integral points.

If $W = \nu X$ is a different birational model, then flop functor $\Psi_\nu : \mathbf{D}^0 W \to \mathbf{D}^0 X$ gives an isomorphism of $\mathbf{K}$-theory $\mathbf{K}W \to \mathbf{K}X$ which maps $\delta_{\nu\mathfrak{J}} \mapsto \delta_{\mathfrak{J}}$. Writing $\beta_i = [\Psi_\nu \mathcal{O}_{C_i}(-1)] \in \mathbf{K}X$ $(i \in \Delta \setminus \nu\mathfrak{J})$ for the images of the simples of $\mathit{per}(\frac{W}{Z})$, we see that $\{\delta_{\mathfrak{J}}, \beta_i \mid i \in \Delta \setminus \mathfrak{J}\}$ is a basis for $\mathbf{K}X$ and the action $\operatorname{Pic} W \circlearrowright \mathbf{K}X$ in this basis is given by a formula analogous to (26). In particular the intersection arrangement is again preserved, and there is an induced action on the set of chambers.

Remark 6.18. For a bounded heart $\mathsf{K} \subset \mathbf{D}^0\mathsf{X}$ and $\mathscr{L} \in \operatorname{Pic} W$, the chamber $\mathscr{L} \cdot \mathbf{C}(\mathsf{K})$ is the heart cone of $\mathscr{L}^\vee \otimes \mathsf{K}$ as should be expected by the contragradience of the dual representation. Thus, strictly speaking, one should declare the action of $\operatorname{Pic} W$ on t-structures to be $\mathscr{L} \cdot \mathsf{K} = \mathscr{L}^\vee \otimes \mathsf{K}$.

The following preliminary lemma shows that nef line bundles play well with the partial order on $\operatorname{Cham}(\underline{\Delta}, \mathfrak{J})$.

Lemma 6.19. *Given a birational model* $W = \nu\mathsf{X}$ *and nef line bundles* $\mathscr{L}, \mathscr{L}' \in \operatorname{Pic} W$, *we have* $\mathscr{L} \cdot C_{\mathfrak{J}}^+ \leqslant \mathscr{L}' \cdot C_{\mathfrak{J}}^+$ *in* $\operatorname{Cham}(\underline{\Delta}, \mathfrak{J})$ *if and only if* $\deg \mathscr{L} \leqslant \deg \mathscr{L}'$.

Proof. If $\mathscr{L} \in \operatorname{Pic}^+ W$ is has degree given by the integers $d_i = (\mathscr{L} \cdot C_i)$ $(i \in \Delta \setminus \nu\mathfrak{J})$, then note that the chambers $C_{\mathfrak{J}}^+$ and $\mathscr{L} \cdot C_{\mathfrak{J}}^+$ are separated by the hyperplane $\{\beta_i - n\delta_{\mathfrak{J}} = 0\}$ if and only if $0 < n \leqslant d_i$.

Now for $\mathscr{L}, \mathscr{L}' \in \operatorname{Pic}^+ W$, we have $\mathscr{L} \cdot C_{\mathfrak{J}}^+ \leqslant \mathscr{L}' \cdot C_{\mathfrak{J}}^+$ if and only if every hyperplane separating $C_{\mathfrak{J}}^+$ and $\mathscr{L} C_{\mathfrak{J}}^+$ also separates $C_{\mathfrak{J}}^+$ and $\mathscr{L}' C_{\mathfrak{J}}^+$. It follows that we necessarily have $\deg \mathscr{L} \leqslant \deg \mathscr{L}'$ if that is the case.

Conversely suppose $\deg \mathscr{L} \leqslant \deg \mathscr{L}'$ and positive real root α lies in $[\mathscr{L}' \cdot C_{\mathfrak{J}}^+ \geqslant 0]$, i.e. for every $\theta \in C_{\mathfrak{J}}^+$ we have $\theta(\alpha) \geqslant 0$ and $\theta(\mathscr{L}' \otimes \alpha) \geqslant 0$. But writing α in terms of the basis $\{\delta_{\mathfrak{J}}, \beta_j \mid j \in \Delta \setminus \nu\mathfrak{J}\}$ and noting $\deg \mathscr{L}' \geqslant \deg \mathscr{L} \geqslant \underline{0}$, we have $\mathscr{L} \otimes \alpha = \alpha + p \cdot \delta_{\mathfrak{J}}$ and $\mathscr{L}' \otimes \alpha = \alpha + q \cdot \delta_{\mathfrak{J}}$ for integers $0 \leqslant p \leqslant q$. It follows that we also have $\theta(\mathscr{L} \otimes \alpha) \geqslant 0$ for every $\theta \in C_{\mathfrak{J}}^+$, i.e. α also lies in $[\mathscr{L} \cdot C_{\mathfrak{J}}^+ \geqslant 0]$. Thus $[\mathscr{L}' \cdot C_{\mathfrak{J}}^+ \geqslant 0] \subseteq [\mathscr{L} \cdot C_{\mathfrak{J}}^+]$, so that $\mathscr{L}' \cdot C_{\mathfrak{J}}^+ \geqslant \mathscr{L} \cdot C_{\mathfrak{J}}^+$ as required. □

Actions on modifying modules. Given a birational model $\pi : W \to Z$ and a line bundle $\mathscr{L} \in \operatorname{Pic} W$, the pushforward $\pi_*\mathscr{L}$ gives a reflexive R-module and this determines an injective homomorphism $\operatorname{Pic} W \to \operatorname{Cl}(R)$ from the Picard group of W to the class group of R. It is clear if W' is another birational model, then the inclusions of $\operatorname{Pic} W$ and $\operatorname{Pic} W'$ into $\operatorname{Cl}(R)$ are compatible with the natural isomorphism $\operatorname{Pic} W \cong \operatorname{Pic} W'$, and we write $\operatorname{cl}(R) \subseteq \operatorname{Cl}(R)$ for the common image of all such inclusions. We remark that the equality $\operatorname{cl}(R) = \operatorname{Cl}(R)$ holds if and only if X (and hence every $W \in \operatorname{Bir}\left(\frac{X}{Z}\right)$) is a minimal model of Z, this follows from [IW, proposition 9.1] and relies on the assumption that the singularity of Z is isolated.

Iyama and Wemyss [IW, §9.4] extensively study the action of $\operatorname{cl}(R)$ on the set $\operatorname{MM}^N(R)$, given by

$$L \cdot M = (L^* \otimes_R M)^{**}$$

for $L \in \operatorname{cl}(R)$, $M \in \operatorname{MM}^N(R)$, and $(-)^* = \operatorname{Hom}(-, R)$. In particular they show that the action is compatible with that on the intersection arrangement under the natural bijection $\mathbf{C} : \operatorname{MM}^N(R) \to \operatorname{Cham}(\underline{\Delta}, \mathfrak{J})$ given by $\mathbf{C}(\nu N) = \nu C_{\mathfrak{J}}^+$, we translate the result into a form suitable for our purposes [see also HW23, lemma 7.2].

Proposition 6.20. *If* $\pi : W \to Z$ *is a birational model of* X, *then for every* $\mathscr{L} \in \operatorname{Pic} W$ *and* $M \in \operatorname{MM}^N(R)$ *we have*

$$\mathscr{L} \cdot \mathbf{C}(M) = \mathbf{C}(\pi_*\mathscr{L} \cdot M).$$

Proof. It suffices to consider $W = X$, and to prove the statement for the line bundles $\mathscr{L}_i = (\det \mathcal{N}_i)^\vee$ $(i \in \Delta \setminus \mathfrak{J})$ which generate $\operatorname{Pic} X$. Further, we may assume the basic modifying module N is *maximal modifying*, i.e.

$$\operatorname{add}(N) = \left\{ M \in \mathit{mod}\, R \text{ reflexive} \mid \operatorname{Ext}^1(N, M) = \operatorname{Ext}^1(M, N) = 0 \right\}.$$

Indeed if not, then note that by [IW, theorem 9.1(1)] and [IW14, corollary 4.18] there is a modifying module N^c such that $N \oplus N^c$ is maximal modifying. This choice can clearly be made such that $N \oplus N^c$ is basic, and further [IW, theorem 9.5] shows that this $N \oplus N^c$ has at most $|\underline{\Delta}|$ indecomposable summands so we can index the summands of N^c as $N^c = \oplus_{i \in \mathfrak{I}} N_i$ for some $\mathfrak{I} \subset \mathfrak{J}$. Then we can work with the pair $(N \oplus N^c, \mathfrak{J} \setminus \mathfrak{I})$ instead of $(N, \mathfrak{J})$ in what follows, using [IW, theorem 8.15] to translate the results back to N.

Writing $\mathit{modif}(R)$ for the category of modifying R-modules, Iyama–Wemyss use the Auslander–McKay correspondence to define the composite map

$$\operatorname{ind} : \mathit{modif}(R) \xrightarrow{[\operatorname{Hom}_R(N,-)]} \mathbf{K}_{\text{split}}(\mathit{proj}\,\Lambda) \otimes \mathbb{R} \xrightarrow{\sim} \Theta(\underline{\Delta}, \mathfrak{J})$$

where simple–projective duality is used to identify $\mathbf{K}_{\text{split}}(\mathit{proj}\Lambda)\otimes\mathbb{R}$ with $\operatorname{Hom}(\mathbf{K}\Lambda,\mathbb{R})\cong\Theta(\underline{\Delta},\mathfrak{J})$. In particular, $\{\operatorname{ind}N_i \mid i\in\underline{\Delta}\setminus\mathfrak{J}\}$ forms the dual basis to $\{\alpha_i \mid i\in\underline{\Delta}\setminus\mathfrak{J}\}$.

Evidently the cone $\mathbf{C}(N)=C^+_{\mathfrak{J}}$ is generated by the vectors $\operatorname{ind}(N_i)$ corresponding to the indecomposable summands of N, and from lemma 4.18 and the discussion preceding its proof it follows that the analogous statement remains true for all $\nu N\in\mathrm{MM}^N(R)$.

Now [IW, proposition 9.10 (1) and theorem 9.23 (2)] show that for any $M\in\mathit{modif}(R)$ we have

$$\begin{aligned}\operatorname{ind}(\pi_*\mathscr{L}_i\cdot M)&=\operatorname{ind}(\det N_i\otimes_R M)^{**}\\&=\operatorname{ind}M+\delta_{\mathfrak{J}}(\operatorname{ind}M)\cdot(\operatorname{ind}(\det N_i)-\operatorname{ind}R)\\&=\operatorname{ind}M+\delta_{\mathfrak{J}}(\operatorname{ind}M)\cdot(\operatorname{ind}N_i-\operatorname{rk}N_i\cdot\operatorname{ind}N_0)\\&=\operatorname{ind}M\circ(\mathscr{L}_i\otimes -)\end{aligned}$$

where the penultimate equality uses the identity $\operatorname{ind}N_i-\operatorname{ind}(\det N_i)=(\operatorname{rk}N_i-1)\cdot\operatorname{ind}R$ which can be deduced, for example, from [IW, corollary 9.22]. Thus $\operatorname{ind}(\pi_*\mathscr{L}_i\cdot M)=\mathscr{L}_i\cdot\operatorname{ind}M$ for every $M\in\mathit{modif}(R)$, and the result follows. □

In what follows we suppress the map π_* from notation, directly considering the action $\operatorname{Pic}W\circlearrowright\mathrm{MM}^N R$. Given $M\in\mathrm{MM}^N R$ and $\mathscr{L}\in\operatorname{Pic}W$, the action gives a natural isomorphism of algebras $\varepsilon:\operatorname{End}_R(\mathscr{L}M)\xrightarrow{\sim}\operatorname{End}_R M$ and thus an equivalence

$$\varepsilon:\mathbf{D}^{\mathrm{fl}}(\operatorname{End}_R\mathscr{L}M)\to\mathbf{D}^{\mathrm{fl}}(\operatorname{End}_R M)$$

which in particular identifies the standard hearts, i.e. $\varepsilon(H)=H$.

For convenience write $\mathscr{L}\cdot\operatorname{End}_R M=\operatorname{End}_R(\mathscr{L}M)$, in particular $\mathscr{L}\Lambda$ is the endomorphism algebra of $\mathscr{L}N$.

Actions on t-structures. Given $\mathscr{L}\in\operatorname{Pic}W$, we examine the intermediacy of $\mathscr{L}^\vee\otimes H$ (with respect to H) by comparing it with the tautologically intermediate heart associated with $\mathscr{L}N\in\mathrm{MM}^N R$, i.e. the image of the Brenner–Butler map

$$\mathit{flmod}(\mathscr{L}\Lambda)\xrightarrow{\mathbf{R}\mathrm{Hom}(\mathrm{Hom}_R(\mathscr{L}\cdot N,N),-)}\mathbf{D}^{\mathrm{fl}}\Lambda.$$

Choosing an atomic sequence of mutations λ from N to $\mathscr{L}\cdot N$, the above map is simply Ψ_λ and the resulting intermediate heart is $\Psi_\lambda H$.

Evidently these agree in **K**-theory, and we now show that these hearts are in fact equal when $\mathscr{L}$ is nef.

Proposition 6.21. *Let $\mathscr{L}\in\operatorname{Pic}W$ be a nef line bundle and λ an atomic path from N to $\mathscr{L}N$. Then the following diagram commutes on objects.*

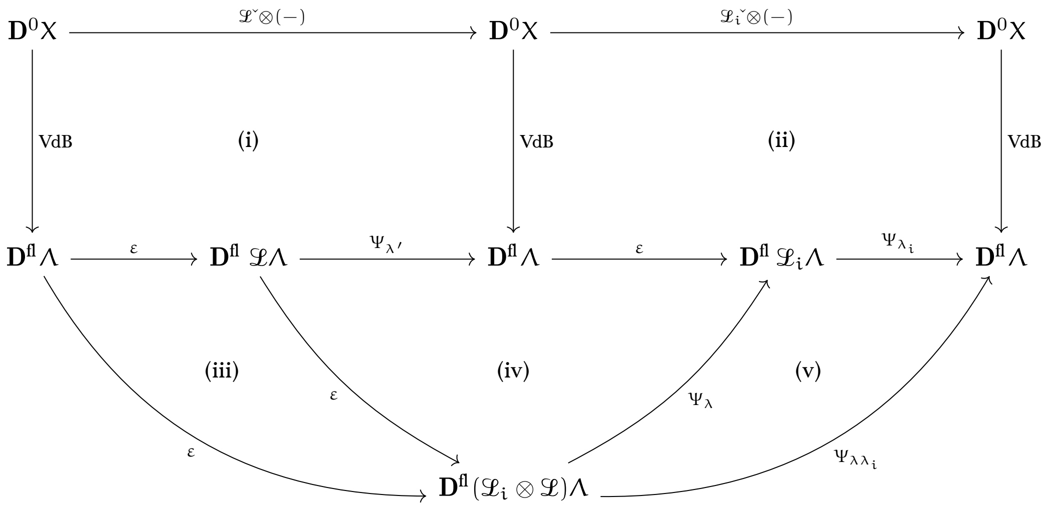
$$\begin{array}{ccccc}\mathbf{D}^0X & & \xrightarrow{\quad\mathscr{L}^\vee\otimes(-)\quad} & & \mathbf{D}^0X\\ \downarrow{\scriptstyle\mathrm{VdB}} & & & & \downarrow{\scriptstyle\mathrm{VdB}}\\ \mathbf{D}^{\mathrm{fl}}\Lambda & \xrightarrow{\ \varepsilon\ } & \mathbf{D}^{\mathrm{fl}}\mathscr{L}\Lambda & \xrightarrow{\ \Psi_\lambda\ } & \mathbf{D}^{\mathrm{fl}}\Lambda\end{array}\tag{27}$$

Proof. First consider the case when $\mathscr{L}$ is a nef bundle on X, i.e. the sequence of flops ν is trivial.

We induct on $\deg\mathscr{L}$, noting that the statement trivially holds for $\mathscr{L}=\mathcal{O}_X$. So suppose the diagram (27) commutes for $\mathscr{L}\in\operatorname{Pic}^+X$, and consider the bundle $\mathscr{L}_i\otimes\mathscr{L}$ for some $i\in\Delta\setminus\mathfrak{J}$.

By lemma 6.19 and lemma 4.16, we can choose positive paths λ, λ_i whose composite $\lambda\lambda_i$ remains atomic such that $\mathscr{L}_i\cdot C^+_{\mathfrak{J}}=\lambda_i C^+_{\mathfrak{J}}$, $(\mathscr{L}_i\otimes\mathscr{L})\cdot C^+_{\mathfrak{J}}=\lambda\lambda_i C^+_{\mathfrak{J}}$. Further since λ is a reduced path from $\mathscr{L}_i\cdot C^+_{\mathfrak{J}}$ to $(\mathscr{L}_i\otimes\mathscr{L})\cdot C^+_{\mathfrak{J}}$, translating it along $\mathscr{L}_i{}^\vee$ gives a reduced (hence atomic) path λ' from $C^+_{\mathfrak{J}}$ to $\mathscr{L}\cdot C^+_{\mathfrak{J}}$.

By proposition 6.20, we have $\mathcal{L}_i N = \lambda N$, $(\mathcal{L}_i \otimes \mathcal{L})N = \lambda\lambda_i N$, and $\mathcal{L} N = \lambda' N$, and each path appearing in these expressions is atomic. Then the corresponding diagram (27) for $\mathcal{L}_i \otimes \mathcal{L}$ is precisely the outer circuit of the diagram below, and it suffices to prove that each component circuit (i)–(v) commutes.



The pentagon (i) commutes by induction hypothesis, and (ii) is precisely the diagram shown to commute in [HW23, theorem 7.4]. The triangle (iii) commutes since there is a natural isomorphism of functors

$$(\pi_* \mathcal{L}_i{}^{\vee} \otimes_R -)^{**} \circ (\pi_* \mathcal{L}^{\vee} \otimes_R -)^{**} \quad \cong \quad (\pi_* (\mathcal{L}_i \otimes \mathcal{L})^{\vee} \otimes_R -)^{**},$$

so that the isomorphism of algebras $\Lambda \xrightarrow{\varepsilon} (\mathcal{L}_i \otimes \mathcal{L})\Lambda$ is the composite $\Lambda \xrightarrow{\varepsilon} \mathcal{L}\Lambda \xrightarrow{\varepsilon} (\mathcal{L}_i \otimes \mathcal{L})\Lambda$. The commutativity of (iv) is the content of [HW23, lemma 7.3], while the triangle (v) commutes by theorem 4.6 since $\lambda\lambda_i$ is an atomic path. Thus the whole diagram commutes as required.

Now suppose we are in the general case, i.e. $\mathcal{L}$ is a nef bundle on the flop $W = \nu X$ and λ is an atomic path from N to $\mathcal{L} N$. Write $M = \nu N$ for the modifying R-module generator associated to W, and choose an atomic path λ' from M to $\mathcal{L} M$. Further we may assume the sequence of flops ν is atomic, so that translating ν by $\mathcal{L}$ as above gives an atomic path ν' from $\mathcal{L} N$ to $\mathcal{L} M$.

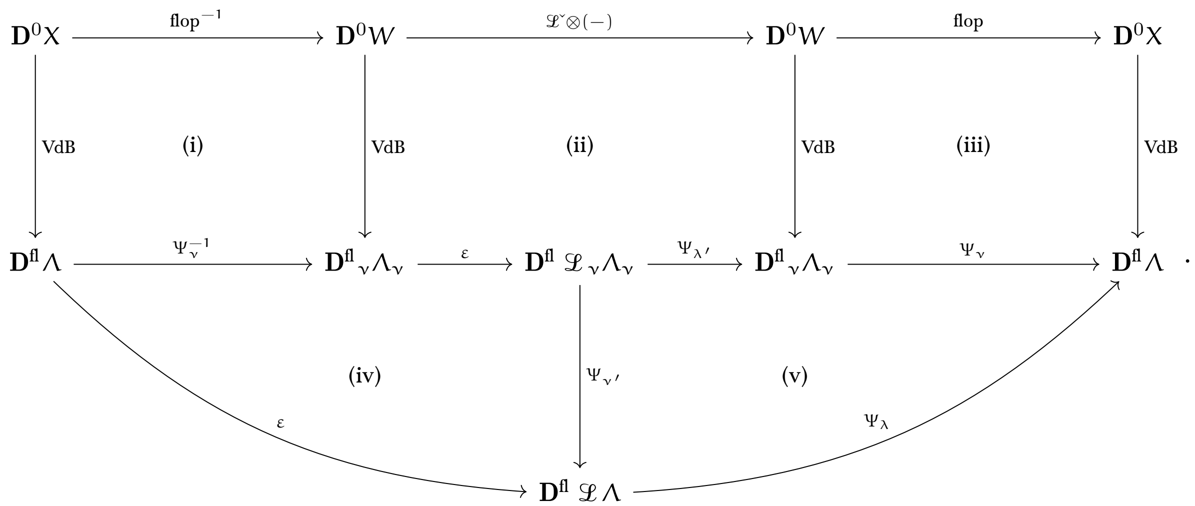
Observe that the composite paths $\lambda'\nu$ and $\nu'\lambda$ are both atomic– indeed if λ' and ν both crossed some root hyperplane $\{\alpha = 0\}$ determined by $\alpha \in \mathrm{Root}^+(\underline{\Delta}, \mathfrak{J})$, then in particular we have $\alpha \in [\nu C^+_{\mathfrak{J}} \leqslant 0]$ and hence $\{\alpha = 0\}$ passes through the ray

$$C^+_{\mathfrak{J}} \cap \nu C^+_{\mathfrak{J}} = \bigcap_{i \in \Delta \setminus \nu\mathfrak{J}} \{\beta_i = 0\},$$

where $\beta_i = [\Psi_\nu S_i]$. Thus $\alpha = \sum_{i \in \Delta \setminus \nu\mathfrak{J}} n_i \beta_i$ for some tuple of non-positive integers n_i.

Since λ' also crosses the given hyperplane, we have $\alpha \in [\mathcal{L}(\nu C^+_{\mathfrak{J}}) \geqslant 0]$ and therefore $\mathcal{L}^{\vee} \otimes \alpha \in [\nu C^+_{\mathfrak{J}} \geqslant 0]$. But this is impossible since $\mathcal{L}$ is nef and hence $\mathcal{L}^{\vee} \otimes \alpha$ is of the form $\sum n_i \beta_i + p\delta_{\mathfrak{J}}$ for some $p \leqslant 0$. Similar reasoning shows the hyperplanes crossed by λ and ν' are distinct.

We can thus construct a diagram



The squares (i) and (iii) commute by theorem 5.4, while the circuit (ii) commutes from the above discussion since $\mathscr{L}$ is a nef bundle on W (the trivial flop of W). The commutativity of (iv) is again the content of [HW23, lemma 7.3], while (v) commutes by using theorem 4.6 and noting that $\mu'\lambda$ and $\lambda'\mu$ are both atomic paths such that $\mu'\lambda N = \lambda'\mu N (= \mathscr{L} M)$. Thus the whole diagram commutes, and reading the outer circuit yields the required diagram (27). □

Intermediacy of twists. We now prove parts (1), (4), (2), and (3) of theorem 6.3 in that order, noting that each successive proof relies on the previous ones.

Proof of theorem 6.3 (1). If $\mathscr{L} \in \operatorname{Pic} W$ is nef, then proposition 6.21 shows that the heart $\mathscr{L}^\vee \otimes H$ coincides with $\Psi_\lambda H$ for some atomic path λ, i.e. lies in $\operatorname{tilt}^+(H)$. In particular, it is intermediate with respect to H.

On the other hand given any $\mathscr{L} \in \operatorname{Pic} W$, we may express it as a difference of nef bundles $\mathscr{L} = \mathscr{L}_+ \otimes \mathscr{L}_-^\vee$ for some $\mathscr{L}_+, \mathscr{L}_- \in \operatorname{Pic}^+ W$. If $\mathscr{L}^\vee \otimes H$ lies in $H[-1,0]$, then the inequality $H \geqslant \mathscr{L}^\vee \otimes H$ implies the inequality $\mathscr{L}_-^\vee \otimes H \geqslant \mathscr{L}_+^\vee \otimes H$ in $\operatorname{tilt}^+(H)$ and thus considering heart cones gives $\mathscr{L}_- \cdot C_{\mathfrak{J}}^+ \leqslant \mathscr{L}_+ \cdot C_{\mathfrak{J}}^+$ in $\operatorname{Cham}(\underline{\Delta}, \mathfrak{J})$ by theorem 4.23. Then lemma 6.19 shows $\deg \mathscr{L}_- \leqslant \deg \mathscr{L}_+$, i.e. $\mathscr{L}$ is necessarily nef.

Thus $\mathscr{L}^\vee \otimes H$ is intermediate with respect to H if and only if $\mathscr{L}$ is nef. One can likewise show that the nefness of $\mathscr{L}$ is also equivalent to the intermediacy of $\mathscr{L} \otimes H[-1]$, by first inverting the diagram (27) to show $\mathscr{L} \otimes H[-1] = \Phi_{\overline{\lambda}} H[-1]$ for some $\mathfrak{J}$-path $\overline{\lambda}$ and then proceeding as above.

Lastly we show $\mathscr{L}^\vee \otimes H > \Psi_\nu \operatorname{coh} W$ whenever $\mathscr{L} \in \operatorname{Pic}^+ W$. Writing $\beta_i = [\Psi_\nu \mathcal{O}_{C_i}(-1)]$ for $i \in \Delta \setminus \nu\mathfrak{J}$, consider the vector $\delta^* \in \Theta(\underline{\Delta} \setminus \mathfrak{J})$ determined as

$$\delta_{\mathfrak{J}}^*(\delta_{\mathfrak{J}}) = 1, \qquad \delta_{\mathfrak{J}}^*(\beta_i) = 0 \quad \text{for all } i \in \Delta \setminus \nu\mathfrak{J}.$$

One checks that $\delta_{\mathfrak{J}}^*(\alpha_i) = 0$ for all $i \in \Delta \setminus \mathfrak{J}$, so we have $\delta_{\mathfrak{J}}^* \in C_{\mathfrak{J}}^+$ and thus the vector $\mathscr{L} \cdot \delta_{\mathfrak{J}}^*$ lies in $\mathscr{L} \cdot C_{\mathfrak{J}}^+ = \mathbf{C}(\mathscr{L}^\vee \otimes H)$. Further since $\mathscr{L}$ is nef, we see that $\mathscr{L} \cdot \delta_{\mathfrak{J}}^* = \delta_{\mathfrak{J}}^* + \theta^0$ for some $\theta^0 \in \nu C_{\mathfrak{J}}^0$. It follows that we have

$$\begin{aligned} \mathscr{L}^\vee \otimes H[1] \cap H \;\subseteq\; H^{\mathrm{tr}}(\mathscr{L} \cdot \delta_{\mathfrak{J}}^*) &= \{h \in H \mid \mathscr{L} \cdot \delta_{\mathfrak{J}}^*[f] \leqslant 0 \text{ for all factors } h \twoheadrightarrow f\} \\ &= \{h \in H \mid \theta^0[f] \leqslant -\delta_{\mathfrak{J}}^*[f] \text{ for all factors } h \twoheadrightarrow f\} \\ &\subseteq \{h \in H \mid \theta^0[f] < 0 \text{ for all non-zero factors } h \twoheadrightarrow f \neq 0\} \\ &= H_{\mathrm{tr}}(\theta_0) \;\subseteq\; \Psi_\nu \operatorname{coh} W[1] \cap H \end{aligned}$$

and thus $\mathscr{L}^\vee \otimes H \geqslant \Psi_\nu \operatorname{coh} W$, with the inequality being necessarily strict since $\Psi_\nu \operatorname{coh} W$ is not algebraic. The proof of the corresponding statement for $\mathscr{L} \otimes H[-1]$ is analogous. □

Proof of theorem 6.3 (4). Given birational models $W = \nu X$ and $W' \in \nu' X$, we have seen that the action of any $\mathcal{L} \in \operatorname{Pic} W$ on $\mathbf{K}X$ coincides with that of its proper transform $\mathcal{L}' \in \operatorname{Pic} W'$. Thus in particular $\mathcal{L}^\vee \cdot C^+_{\mathfrak{J}} = \mathcal{L}'^\vee \cdot C^+_{\mathfrak{J}}$, i.e. $\mathbf{C}(\mathcal{L} \otimes H) = \mathbf{C}(\mathcal{L}' \otimes H)$.

Now if $\mathcal{L} \otimes H$ and $\mathcal{L}' \otimes H$ are intermediate with respect to H, then (the proof of) theorem 6.3 (1) shows that they both lie in $\operatorname{tilt}^+(H)$. But the assignment $\mathbf{C} : \operatorname{tilt}^+(H) \to \operatorname{Cham}(\underline{\Delta}, \mathfrak{J})$ is bijective, so $\mathcal{L}^\vee \otimes H = \mathcal{L}'^\vee \otimes H$.

One can analogously show the hearts $\mathcal{L} \otimes H[-1]$ and $\mathcal{L}' \otimes H[-1]$ are equal when intermediate. □

Proof of theorem 6.3 (2). Suppose σ and W are as given, and consider a line bundle $\mathcal{L} \in \operatorname{Pic} W \cap \sigma$. If υX is another σ-positive birational model (i.e. $\sigma \subset \upsilon C^0_{\mathfrak{J}}$), then the proper transform of $\mathcal{L}$ is nef on υX and thus theorem 6.3 (1) and (4) give us $\mathcal{L}^\vee \otimes H > \Psi_\upsilon \operatorname{coh}(\upsilon X)$. Considering all such inequalities together, we obtain

$$\inf \{\mathcal{L}^\vee \otimes H \mid \mathcal{L} \in \operatorname{Pic} W \cap \sigma\} \;\geqslant\; \sup \left\{\Psi_\upsilon \operatorname{coh}(\upsilon X) \mid \sigma \subset \upsilon C^0_{\mathfrak{J}}\right\}.$$

By theorem 5.28 the supremum in the right-hand side of the above inequality is precisely $H^{tt}(\sigma)$.

Conversely suppose $h \in H$ lies in the torsion-free class associated to $\inf \{\mathcal{L}^\vee \otimes H \mid \mathcal{L} \in \operatorname{Pic} W \cap \sigma\}$, i.e. $h \in \mathcal{L}^\vee \otimes H$ for every $\mathcal{L} \in \operatorname{Pic} W \cap \sigma$. Pick a line bundle $\mathcal{L}_0$ that lies *generically* in $\operatorname{Pic} W \cap \sigma$, i.e. $\mathcal{L}_0$ does not lie in $\operatorname{Pic} W \cap \sigma'$ for any proper face $\sigma' \subset \sigma$. Thus $\mathcal{L}_0$ determines a generic vector $\theta_0 \in \sigma$ under the identification of proposition 5.5. Considering the vector $\theta = \mathcal{L}_0^{\otimes n} \cdot \delta^*_{\mathfrak{J}}$ where $n > 0$ is some integer and $\delta^*_{\mathfrak{J}} \in \Theta(\underline{\Delta} \setminus \mathfrak{J})$ is the vector given by (20), one calculates $\theta = \delta^*_{\mathfrak{J}} + n \cdot \theta_0$

Moreover θ clearly lies in $\mathcal{L}_0^{\otimes n} \cdot C^+_{\mathfrak{J}} = \mathbf{C}((\mathcal{L}_0^{\otimes n})^\vee \otimes H)$, and thus we have $h \in H_{tf}(\theta)$ i.e. $\delta^*_{\mathfrak{J}}[s] + n \cdot \theta_0[s] > 0$ for every non-zero sub-object $s \hookrightarrow h$. But $n > 0$ was arbitrary and $\delta^*_{\mathfrak{J}}[s] \geqslant 0$, so we must in fact have $\theta_0[s] \geqslant 0$ for every sub-object $s \hookrightarrow h$. In other words $h \in H^{tf}(\theta_0)$, and thus we obtain the converse inequality $\inf \{\mathcal{L}^\vee \otimes H \mid \mathcal{L} \in \operatorname{Pic} W \cap \sigma\} \leqslant H^{tt}(\theta_0) = H^{tt}(\sigma)$ as required.

The statement for $H_{tt}(\sigma)$ can be proved analogously. □

Before proceeding, we make an observation. Given $\sigma \in \operatorname{Arr}(\Delta, \mathfrak{J})$ and a σ-positive birational model W as above, every $\mathcal{L}' \in \operatorname{Pic} W \cap \sigma$ can be bound by some power of a line bundle $\mathcal{L}_0 \in \operatorname{Pic} W$ that lies generically in $\sigma \cap \operatorname{Pic} W$. That is to say for any $\mathcal{L}' \in \operatorname{Pic} W \cap \sigma$ there is an integer $n > 0$ such that $\deg \mathcal{L}_0^{\otimes n} \geqslant \deg \mathcal{L}'$. Indeed, one can choose n to be the maximum coordinate of the vector $\deg \mathcal{L}'$.

Combining this with corollary 6.4 (which only relies on part (1) of theorem 6.3), we see that for any $\mathcal{L}' \in \operatorname{Pic} W \cap \sigma$ there is an integer $n > 0$ such that $\mathcal{L}'^\vee \otimes H \geqslant (\mathcal{L}_0^\vee)^{\otimes n} \otimes H$. Thus $H^{tt}(\sigma)$, which by theorem 6.3 (2) is the infimum of the poset $\{\mathcal{L}'^\vee \otimes H \mid \mathcal{L}' \in \operatorname{Pic} W \cap \sigma\}$, is also the infimum of the *chain* $H \geqslant \mathcal{L}_0^\vee \otimes H \geqslant (\mathcal{L}_0^\vee)^{\otimes 2} \otimes H \geqslant \dots$.

Likewise, $H_{tt}(\sigma)$ is the supremum of the chain $H[-1] \leqslant \mathcal{L}_0 \otimes H[-1] \leqslant \mathcal{L}_0^{\otimes 2} \otimes H \leqslant \dots$.

This greatly simplifies the following proof, since the supremum of a chain of torsion(-free) classes in H is simply the nested union.

Proof of theorem 6.3 (3). Suppose $\mathcal{L}' \in \operatorname{Pic} W'$ satisfies $H \geqslant \mathcal{L}'^\vee \otimes H > \Psi_\nu \operatorname{coh} W$, and consider the torsion class $T = \mathcal{L}'^\vee \otimes H[1] \cap H$.

Picking a line bundle $\mathcal{L}_0 \in \operatorname{Pic} W$ that lies generically in $\operatorname{Pic} W \cap \nu C^0_{\mathfrak{J}}$, the above discussion shows we have

$$\begin{aligned} T &\subseteq \Psi_\nu \operatorname{coh} W[1] \cap H \\ &= \inf \left\{(\mathcal{L}_0^\vee)^{\otimes n} \otimes H \mid n \geqslant 0\right\}[1] \cap H \\ &= \bigcup_{n \geqslant 0} (\mathcal{L}_0^\vee)^{\otimes n} \otimes H[1] \cap H. \end{aligned}$$

Now the torsion class T is finitely generated (for example by the finite set of brick labels in the interval $[\mathcal{L}'^\vee \otimes H, H] \subset \operatorname{tilt}^+(H)$), and each generator $t_i \in T$ lies in some torsion class $(\mathcal{L}_0^\vee)^{\otimes n_i} \otimes H[1] \cap H$ in the above nested union. Taking n to be the maximum of such n_i (taken over a chosen generating set $\{t_1, \dots, t_k\} \subset T$), we thus see that each t_i lies in $(\mathcal{L}_0^\vee)^{\otimes n} \otimes H[1] \cap H$, and hence so does T.

Replacing $\mathscr{L}_0$ with $\mathscr{L}_0^{\otimes n}$ if necessary, we thus have the inequality of intermediate hearts $\mathscr{L}' \otimes \mathsf{H} \geqslant \mathscr{L}_0^{\vee} \otimes \mathsf{H}$. One can analogously show that if $\mathscr{L}' \otimes \mathsf{H}[-1]$ lies in $[\mathsf{H}[-1], \Psi_\nu \, \mathit{coh}\, W]$, then $\mathscr{L}' \otimes \mathsf{H}[-1] \leqslant \mathscr{L}_0 \otimes \mathsf{H}[-1]$ for some $\mathscr{L}_0 \in \operatorname{Pic}^+ W$.

In either case, theorem 4.23 gives the inequality of chambers $\mathscr{L}' \cdot C_{\mathfrak{J}}^+ \leqslant \mathscr{L}_0 \cdot C_{\mathfrak{J}}^+$, or equivalently $\mathscr{L} \cdot C_{\mathfrak{J}}^+ \leqslant \mathscr{L}_0 \cdot C_{\mathfrak{J}}^+$ where $\mathscr{L} \in \operatorname{Pic} W$ is the proper transform of $\mathscr{L}'$. That is to say, any root half-space which contains both $C_{\mathfrak{J}}^+$ and $\mathscr{L}_0 \cdot C_{\mathfrak{J}}^+$ must also contain $\mathscr{L} \cdot C_{\mathfrak{J}}^+$. We use this convex-geometric condition to show $\mathscr{L}$ must be nef, i.e. $d_i = (\mathscr{L} \cdot C_i) \geqslant 0$ for each exceptional curve $C_i \subset W$.

To see this, consider the root $\beta_i = [\Psi_\nu \mathcal{O}_{C_i}(-1)]$. If β_i is a positive root, then $C_{\mathfrak{J}}^+ \subseteq \{\delta_{\mathfrak{J}} \geqslant \beta_i \geqslant 0\}$, and hence

$$\begin{aligned} \mathscr{L} \cdot C_{\mathfrak{J}}^+ &\subseteq \mathscr{L} \cdot \{\delta_{\mathfrak{J}} \geqslant \beta_i \geqslant 0\} \\ &= \{\mathscr{L}^{\vee} \otimes \delta_{\mathfrak{J}} \geqslant \mathscr{L}^{\vee} \otimes \beta_i \geqslant 0\} \\ &= \{\delta_{\mathfrak{J}} \geqslant \beta_i - d_i \delta_{\mathfrak{J}} \geqslant 0\} \\ &= \{(d_i + 1)\delta_{\mathfrak{J}} \geqslant \beta_i \geqslant d_i \delta_{\mathfrak{J}}\}. \end{aligned}$$

But by nef-ness of $\mathscr{L}_0$, both $C_{\mathfrak{J}}^+$ and $\mathscr{L}_0 \cdot C_{\mathfrak{J}}^+$ lie in the half space $\{\beta_i \geqslant 0\}$, hence so does $\mathscr{L} \cdot C_{\mathfrak{J}}^+$. In particular, the intersection $\{\beta_i \geqslant 0\} \cap \{(d_i + 1)\delta_{\mathfrak{J}} \geqslant \beta_i \geqslant d_i \delta_{\mathfrak{J}}\}$ is non-empty, and hence $d_i \geqslant 0$.

Likewise if β_i is a negative root, then one sees $C_{\mathfrak{J}}^+ \subseteq \{0 \geqslant \beta_i \geqslant -\delta_{\mathfrak{J}}\}$ and thus $\mathscr{L} \cdot C_{\mathfrak{J}}^+ \subseteq \{d_i \delta_{\mathfrak{J}} \geqslant \beta_i \geqslant (d_i - 1)\delta_{\mathfrak{J}}\}$. On the other hand $C_{\mathfrak{J}}^+$ and $\mathscr{L}_0 \cdot C_{\mathfrak{J}}^+$ (and hence also $\mathscr{L} \cdot C_{\mathfrak{J}}^+$) lie in the half-space $\{\beta_i \geqslant -\delta_{\mathfrak{J}}\}$ so that the intersection $\{d_i \delta_{\mathfrak{J}} \geqslant \beta_i \geqslant (d_i - 1)\delta_{\mathfrak{J}}\} \cap \{\beta_i \geqslant -\delta_{\mathfrak{J}}\}$ is non-empty. This again shows $d_i \geqslant 0$, and hence $\mathscr{L}$ is nef as required. □

The proofs of propositions 6.1 and 6.2 and theorem 6.5 now follow from convex-geometric arguments.

Proof of proposition 6.1. Given the line bundle $\mathscr{L} = \mathscr{L}_i^{\otimes n}$ ($i \in \Delta \setminus \mathfrak{J}$, $n > 0$), theorem 6.3 (1) shows that $\mathscr{L}^{\vee} \otimes \mathsf{H}$ lies in $\operatorname{tilt}^+(\mathsf{H})$ and is in particular intermediate with respect to H. Further since it is an Artinian tilt, proposition 2.6 shows the torsion class $\mathscr{L}^{\vee} \otimes \mathsf{H}[1] \cap \mathsf{H} \in \operatorname{tors}(\mathsf{H})$ is generated by the semibrick

$$\begin{aligned} \mathsf{S} &= \{b \in \mathsf{H} \mid b[-1] \text{ is a simple object of } \mathscr{L}^{\vee} \otimes \mathsf{H}\} \\ &= \{b \in \mathsf{H} \mid b \text{ is the brick label of a covering relation } \mathsf{K} \gtrdot \mathscr{L}^{\vee} \otimes \mathsf{H} \text{ in } \operatorname{tilt}(\mathsf{H})\}. \end{aligned}$$

In particular, the size of S is equal to the number of hearts in $\operatorname{tilt}(\mathsf{H})$ covered by $\mathscr{L}^{\vee} \otimes \mathsf{H}$.

By corollary 4.7 and theorem 4.23 this is the number of covering relations of the form $\sigma \lessdot \mathscr{L}\, C_{\mathfrak{J}}^+$ in $\operatorname{Cham}(\underline{\Delta}, \mathfrak{J})$. Such a covering relation arises precisely from a positive root $\alpha \in [\mathscr{L}\, C_{\mathfrak{J}}^+ \leqslant 0]$ such that the hyperplane $\{\alpha = 0\}$ contains a simple wall of $\mathscr{L}\, C_{\mathfrak{J}}^+$. But the simple walls of $\mathscr{L}\, C_{\mathfrak{J}}^+$ are defined by the positive roots

$$n\delta_{\mathfrak{J}} - \alpha_i, \qquad \alpha_j \geqslant 0 \quad (j \in \Delta \setminus \mathfrak{J},\ j \neq i), \qquad \alpha_0 + n\delta_i \cdot \delta_{\mathfrak{J}}$$

and of these only $n\delta_{\mathfrak{J}} - \alpha_i$ lies in $[\mathscr{L}\, C_{\mathfrak{J}}^+ \leqslant 0]$.

Thus the semibrick S has precisely one element, and examining the simples of $\mathscr{L}^{\vee} \otimes \mathsf{H}$ shows that this element must be $\mathcal{O}_{C_i}(-n-1)[1] = \mathscr{L}^{\vee} \otimes \mathcal{O}_{C_i}(-1)[1]$ as required.

The statement for $\mathscr{L} \otimes \mathsf{H}[-1]$ can be proved analogously. □

Proof of proposition 6.2. Immediate from proposition 6.1 and theorem 6.3 (2). □

Proof of theorem 6.5. We exhibit the proof for $\operatorname{tilt}^+(\mathsf{H})$, with the proof for $\operatorname{tilt}^-(\mathsf{H})$ being analogous. Now given $\mathsf{K} \in \operatorname{tilt}^+(\mathsf{H})$ there is a unique birational model $W = \nu X$ such that

$$\begin{aligned} \mathbf{C}\,\mathsf{K} \;&\subseteq\; \nu C_{\mathfrak{J}}^{0} \;+\; \mathbb{R}_{\geqslant 0} \cdot \delta_{\mathfrak{J}}^{*} \\ &=\; \left\{\theta_0 + t\delta_{\mathfrak{J}}^{*} \mid \theta_0 \in \nu C_{\mathfrak{J}}^{0},\ t \geqslant 0\right\}, \end{aligned}$$

where $\delta_{\mathfrak{J}}^{*}$ is as in (20). Indeed, the half space $\{\delta_{\mathfrak{J}} \geqslant 0\}$ (which contains $\operatorname{Arr}^+(\underline{\Delta}, \mathfrak{J})$) can be written as the union of all such regions. Thus in particular for each $\beta_i = [\Psi_\nu \mathcal{O}_{C_i}(-1)]$ ($i \in \Delta \setminus \nu\mathfrak{J}$), we have $\mathbf{C}\,\mathsf{K} \subseteq \{\beta_i \geqslant 0\}$.

A similar argument involving a decomposition of $\{\delta_{\mathfrak{J}} \geqslant 0\}$ shows that for each $i \in \Delta \setminus \nu\mathfrak{J}$ there is a unique integer d_i such that

$$\mathbf{C}\,K \subseteq \{(d_i+1)\delta_{\mathfrak{J}} \geqslant \beta_i \geqslant d_i\delta_{\mathfrak{J}}\}, \tag{28}$$

and clearly we must have $d_i \geqslant 0$ in this case. Then the dimension vector $(d_i) \in \mathbb{Z}^{|\Delta\setminus\nu\mathfrak{J}|}$ determines line bundles $\mathscr{L}, \mathscr{L}' \in \operatorname{Pic} W$ with degrees (d_i), (d_i+1) respectively, and by construction we have

$$\underline{0} \leqslant \deg \mathscr{L}^\vee < \deg \mathscr{L}'^\vee = \deg \mathscr{L}^\vee - \underline{1}.$$

It remains to show K lies in the interval $[\mathscr{L}'^\vee \otimes H, \mathscr{L}^\vee \otimes H]$ or equivalently, that $\mathscr{L} \cdot C^+_{\mathfrak{J}} \leqslant \mathbf{C}\,K \leqslant \mathscr{L}' \cdot C^+_{\mathfrak{J}}$.

Suppose a root α lies in $[\mathscr{L}\cdot C^+_{\mathfrak{J}} \leqslant 0]$, then by lemma 6.19 we see that α lies in $[\mathscr{L}''\cdot C^+_{\mathfrak{J}} \leqslant 0]$ for every $\mathscr{L}'' \in \operatorname{Pic}^+ W$ of sufficiently large degree. It follows that the cone $\nu C^0_{\mathfrak{J}}$ lies in the half-space $\{\alpha \leqslant 0\}$. Now for any $i \in \Delta \setminus \mathfrak{J}$, the bounds (28) imply that $\mathbf{C}\,K$ lies in the region $\mathscr{L} \cdot \{\beta_i, \delta_{\mathfrak{J}} \geqslant 0\}$, and hence every $\theta \in \mathbf{C}\,K$ can be expressed as $\theta = \mathscr{L} \cdot (\theta_0 + t\delta^*_{\mathfrak{J}})$ for some $\theta_0 \in \nu C^0_{\mathfrak{J}}$, $t \geqslant 0$. The root α is negative on both $\mathscr{L} \cdot \theta_0 \in \mathscr{L} \cdot \nu C^0_{\mathfrak{J}} \subset \nu C^0_{\mathfrak{J}}$ and $\mathscr{L} \cdot \delta^*_{\mathfrak{J}} \in \mathscr{L} \cdot C^+_{\mathfrak{J}}$, and thus $\theta(\alpha) \leqslant 0$ i.e. $\alpha \in [\mathbf{C}\,K \leqslant 0]$.

This shows $[\mathscr{L} \cdot C^+_{\mathfrak{J}} \leqslant 0] \subseteq [\mathbf{C}\,K \leqslant 0]$, i.e. $\mathscr{L} \cdot C^+_{\mathfrak{J}} \leqslant \mathbf{C}\,K$. One analogously argues $[\mathscr{L}' \cdot C^+_{\mathfrak{J}} \geqslant 0] \subseteq [\mathbf{C}\,K \geqslant 0]$ to obtain the desired conclusion. □

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